Symbolical Algebra as a Foundation for Calculus:
D. F. Gregory’s Contribution

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Duncan Farquharson Gregory (1813–1844) was a mathematician and founder of the Cambridge Mathematical Journal. His 1841 Examples of the Processes of the Differential and Integral Calculus was an extensive revision of Peacock’s 1820 textbook of a similar title. Among the new material in Gregory’s version is an exposition of symbolical algebra, prominently featuring the method of “separation of symbols.” We examine Gregory’s career; the mathematicians he influenced; and Servois and Murphy, who influenced him. We consider Gregory’s use of separation of symbols in the Examples and consider whether he believed these techniques to be capable of providing an adequate foundation for calculus. © 2002 Elsevier Science (USA)


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1. INTRODUCTION

For more than a century following the discovery of the calculus, mathematicians were aware that a rigorous foundation, something in the style of Euclid’s *Elements* for example, was lacking. Bishop Berkeley first enunciated the challenge of finding such a foundation in *The Analyst*. During the course of the 18th century, the competing notions of limit and infinitesimal were variously championed by those seeking such a logical grounding. Toward the end of the century a third path emerged. It was Lagrange who first sought to reduce analysis to algebra, by means of power series, and thereby to give the calculus a foundation that depended neither on the limit nor on the infinitesimal.

Neither the upstart algebraic approach nor Leibniz’ infinitesimals would lead to a satisfactory foundation in the 19th century. With hindsight, modern mathematicians would say that Cauchy, by reducing the idea of limit to arithmetical principles based on simple inequalities (the so-called “epsilon–delta” approach), laid the issue to rest. In his 1821 *Cours d’Analyse* and his 1823 *Résumé des Leçons données a L’Ecole Royale Polytechnique sur le Calcul Infinitésimal*, Cauchy used his notion of limit to provide acceptable definitions of continuity and convergence, as well as the first rigorous proofs of the mean value theorem and the fundamental theorem of calculus. How can it make sense, then, even to consider the question of whether Duncan F. Gregory believed in 1841 that algebra might still provide the route to a rigorous foundation of calculus?

Gregory’s lifespan was tragically brief and accordingly he left us with relatively few words. Specifically, he never wrote explicitly on the foundations of calculus, and so his beliefs on the subject must inferred indirectly. The only book-length work he saw published in his lifetime was his 1841 *Examples in the Processes of the Differential and Integral Calculus*. This is not an elementary calculus book, but rather a supplementary text, which presupposes the student has already learned the elementary principles of calculus. It begins with a summary of the elementary formulas of differentiation, not a derivation of them. The text functioned equally well whether the student’s understanding of first principles was algebraic in flavor (e.g., following Peacock’s notes in the English translation of Lacroix’s *Traité élémentaire* [Lacroix 1816]) or used limits (e.g., De Morgan’s calculus text [De Morgan 1842]). Indeed, someone who had learned calculus using the Newtonian doctrine of fluxions would still have benefited from Gregory’s text. Nowhere does Gregory make a pronouncement on whether limits or power series should be used to define the derivative and subsequently be employed in the derivation of elementary differentiation formulas.
At places in his 1841 text, Gregory provides proofs of propositions. However, these propositions do not concern the fundamental matters of the calculus, but deal instead with more advanced differentiation formulas, Taylor series, and techniques used in the solution of differential equations. In these proofs, Gregory uses certain algebraic techniques, referred to contemporarily as “symbolical algebra.” These techniques can be traced back to Lagrange and had evolved considerably since that time. The use of Lagrangian techniques here suggests a Lagrangian conception of first principles as well. In any case, Gregory endorses the work of the mathematicians Servois and Murphy, who clearly believed that algebra is the appropriate logical basis for calculus, and uses his text to promote wider use and acceptance of symbolical algebra.

In this paper we will carefully examine the career and writings of Duncan Farquharson Gregory, teasing out any clues to his conception of a foundation for calculus.

2. BIOGRAPHY

Duncan Farquharson Gregory was born in Edinburgh, Scotland, in April 1813. Secondary sources, such as the Dictionary of Scientific Biography [1972, 5:522], give the date as April 13, although a baptismal record from July of that year specifies April 14. He was the youngest son of 11 children born to James Gregory and his second wife Isabella, nee McLeod.

Duncan was a member of the distinguished Gregory family of Scottish scientists. His elder brother, the chemist William (1802–1858), and his paternal grandfather, John (1724–1773), were professors of medicine at the University of Edinburgh. Duncan’s father James (1753–1821) was professor of medicine at King’s College, Aberdeen, as were James’ uncle and grandfather, both also named James. One generation further back in the line of patrimonial descent was James Gregory (1638–1675), the mathematician.

James the mathematician was the inventor of an early reflecting telescope called the Gregorian telescope, the discoverer of the familiar infinite series expansion of arctangent (discovered independently two years later in 1673 by Leibniz), and the first professor of mathematics at the University of Edinburgh. James also worked on the relationship between the tangent problem and the area problem. James’ nephew David (1659–1708), Duncan’s first cousin twice removed, was also professor of mathematics at Edinburgh. He held the position from 1683 to 1690 and was elected Savilian Professor at Oxford in 1691, owing in no small part to the support of Isaac Newton. David, for his part, was a staunch supporter of Newton in the Newton–Leibniz controversy.

Duncan was tutored solely by his mother until the age of nine, when a part-time private tutor was hired.3 As a child as in adulthood, his interests were varied: history, geography, gardening, astronomy, chemistry, botany, and natural philosophy, in addition to mathematics. At age eight he made an orrery, a model demonstrating the positions and motions of heavenly bodies. He attended the newly opened Edinburgh Academy from 1824 to 1827. The emphasis at this very expensive day school was on classical Greek studies, and there was a Master to provide a model of English as spoken in England; see [Osborne 1967, 211] or [Daiches 1978, 197–198]. He spent the next winter at a private academy in Geneva, where

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3 Except as indicated, the biographical facts about D. F. Gregory are contained in an unsigned obituary in Gentlemen’s Magazine [1844, 21:657], a memoir of Gregory by Robert Leslie Ellis in the Cambridge Mathematical Journal [1845:4], and Venn’s Alumni Cantabrigiensis [Venn 1922].
his talent for mathematics was noticed, and then attended the University of Edinburgh. At Edinburgh, he became a pupil and protégé of William Wallace (1768–1843), working in chemistry and light, as well as mathematics.

In 1833 Gregory matriculated at Trinity College, Cambridge, from which he received his B.A. in 1838. He sat for the tripos, or Mathematical Honours Examination, in 1837. At that time, candidates were ranked on the basis of their performance on the tripos; the top 12 were called wranglers, with the highest scorer named first, or senior wrangler. In the list of honorees at the Bachelor of Arts Commencement of January 21, 1837, Gregory was named fifth wrangler. Fourth place went to 43-year-old George Green of Caius, now remembered for Green’s Theorem. The top three wranglers were all students at St. John’s, the second wrangler being J. J. Sylvester who, because he was Jewish, was not able to receive a degree or be named a fellow.

Assisted by Robert Leslie Ellis (1817–1859), who was to be senior wrangler in 1840, and Samuel S. Greathed (1813–1887), fourth wrangler in 1835, Gregory founded the Cambridge Mathematical Journal (CMJ) in 1837. Gregory was editor and a principal contributor until his death. His contributions to this journal, together with his essay “On the Real Nature of Symbolical Algebra” [Gregory 1840], which was read before the Royal Society of Edinburgh on May 7, 1838, are collected in The Mathematical Writings of Duncan Farquharson Gregory [Gregory 1865], edited by William Walton and published in 1865. In 1838, Gregory was considered for the mathematics chair at the University of Edinburgh, but lost to Philip Kelland (1808–1879) in an especially interesting contest.

Gregory was elected Fellow of Trinity in 1840 and received his Master of Arts the following year. Also in 1841, he published Examples of the Processes of the Differential and Integral Calculus [Gregory 1841a] and became Moderator (principal examiner) of the tripos. That same year, he was courted for a professorship in mathematics at the University of Toronto, but declined. He was also the leading contender for the natural philosophy chair at Glasgow in a search begun in late 1841. However, he died before the chair became vacant in 1846, and William Thompson (the future Lord Kelvin) was named to the position.

As was the custom, after his year-long term as moderator, Gregory continued to participate in the tripos as examiner. In 1842, he began work on A Treatise on the Application of Analysis to Solid Geometry [Gregory 1845], which was published posthumously.

Gregory suffered his “first attack of illness” late in 1842. In a letter to George Boole written from London in June 1843, he spoke of “a severe attack of illness, from which I have not yet recovered” (quoted in [Harley 1866, 442]). Gregory left Cambridge in the spring of 1843, but continued to write until his death on February 23, 1844. Just before his death, he had been working on a paper on the fundamental connection between differential equations and finite difference equations. According to Ellis, it was an “analogy” on which he had worked at great length but on which he had published little. An obituary in stated that “his nervous system was impaired by severe study, inducing bodily disease, which proved fatal” [Gentlemen’s Magazine 1844, 21:657]. In a letter dated October 24, 1843, James Thomson (professor of mathematics at Glasgow University) wrote to his son William about Gregory’s serious illness:

...some of the most eminent medical men in Edin. think Gregory is in a very dangerous state, which is very melancholy. They think there is an internal tumour, which is beyond the power of medicine [J. Thomsons 1843].
Thomson’s comment, the only specific reference to the nature of Gregory’s illness and probable cause of death, is consistent with Ellis’ remark that Gregory was in almost constant pain during the last months of his life [Ellis 1845, 150]. The official death record states that the cause of death was “decline” and that he was buried at Canongate Churchyard, Edinburgh.

As noted above, Gregory was considered for the mathematics chair at the University of Edinburgh. He and Kelland were the leading candidates to replace the very traditional Wallace who was expected to retire in 1838. Wallace was so distrustful of what he considered to be the logically weak analysis (meaning, in this case, algebra) that he fortified his students against it by giving them increased doses of the logically ideal geometry.

Those favoring Kelland, an Anglican cleric, did so because he was completely unfamiliar with Scotland and Scottish education and had an excellent reputation at Cambridge. Kelland’s supporters, including James D. Forbes (1809–1868), the holder of the natural philosophy chair at Edinburgh, thought that, in addition to emphasizing British analysis, he would introduce into Edinburgh the purportedly objective Cambridge system of ranking students numerically.

Those supporting Gregory noted that he was a member of the Gregory family so prominent in Scottish science, had been an outstanding pupil of Wallace, and yet was a success among the Cambridge mathematicians. In their view, he combined knowledge of the new mathematics with a respect for his country’s strong geometric tradition [Davie 1964, 118]. The Gregory side found additional support from the noted philosopher and educational reformer Sir William Hamilton, who encouraged the electors not to “be frightened into voting for a stranger.” Gregory, Hamilton argued, would maintain what was valuable in the Scottish tradition, while Kelland was not qualified to do so. However, Kelland won by a narrow margin and was appointed to the chair in 1838. The next contact between Gregory and Kelland was occasioned by a letter written by William Thomson, the future Lord Kelvin, who was to become Gregory’s most prominent protégé.

3. GREGORY AS MENTOR

To measure Gregory’s true impact on 19th century British mathematics, one must consider his influence as a mentor and as founder and editor of the Cambridge Mathematical Journal, in addition to his original research and expository writing.

In 1837, Kelland published Theory of Heat, in which he criticized certain aspects of Fourier’s mathematics. Three years later, 16-year-old William Thomson, son of James Thomson, read Kelland’s work and realized that the criticism of Fourier was unfounded. Thomson noted that Kelland had erred, at least in part, by not giving sufficient consideration to the interval over which the function was expanded as a Fourier series.

The Lord Kelvin Collection at Cambridge University contains unpublished correspondence among the Thomsons and Gregory discussing William’s rebuttal of Kelland. Included is a draft of a letter [W. Thomson 1841] written by William, but signed “PQR,” containing the rebuttal. James sent his son’s rebuttal to Gregory, who was a personal friend, without naming the author and asked that it be considered for publication in the CMJ. Gregory responded, agreeing with the contents of the paper. However, he explained that the controversial nature of the article suggested that anonymous publication would not be wise [Gregory 1841b]. Eventually, James provided Gregory with the author’s name and sent a copy of the article to
Kelland replied “under a feeling of irritation” and persuaded the anonymous author to make some changes [J. Thomson 1841a]. The article, William’s first, was published in the *Cambridge Mathematical Journal* in 1841, signed PQR, and Kelland’s reputation as a mathematician was seriously damaged [Davie, 1964, 126].

In the autumn of 1841, when William Thomson began his studies at Cambridge, James arranged for his son to meet Gregory. Within a few months William developed a great admiration for Gregory, peppering his letters to his father with phrases like “Gregory says,” “Gregory told me,” and “Gregory thinks” [Smith and Wise 1989, 181].

While Gregory played a significant role in Thomson’s early professional and academic life, we find little evidence that Gregory’s mathematics influenced Thomson’s. Such is not the case, however, with George Boole (1815–1854). Not only was Gregory a mentor to Boole in the early years of his career, but Boole’s mathematics clearly reflects Gregory. This influence is demonstrated by their correspondence (quoted in [Harley 1866, 425–472]) and notes in Boole’s books and papers. Gregory was responsible for Boole’s first published paper (February, 1840), which appeared in the *Cambridge Mathematical Journal* [2, 64–73], to be followed by 10 others.

Gregory’s mentoring of Boole is evidenced in an 1840 letter, praising a later paper Boole had submitted to *CMJ* and explaining the reasons, which had nothing to do with mathematical content, that the paper had been rejected by another journal, offering to rewrite it, and encouraging him in his current research.

Boole’s first submission to the Royal Society, “On a General Method in Analysis,” sent on the advice of Gregory, was awarded the Royal Medal as the “most important unpublished paper in Mathematics, communicated to the Royal Society for insertion in their Transactions” in the 1841–1844 period [Royal Society of London 1844]. In a footnote to the paper, Boole praised Gregory and acknowledged his debt to him:

> Few in so short a life have done so much for science. The high sense which I entertain of his merits as a mathematician, is mingled with feelings of gratitude for much valuable assistance rendered to me in my earlier essays [Boole 1844, 279].

In his 1859 book, *A Treatise on Differential Equations*, Boole wrote in a footnote [Boole 1859, 391, n] of his “brief but valued friendship” with Gregory. In the same note he says that the theorem being cited [Boole 1840] was written and diagrammed by Gregory, using notes from Boole, but that Gregory allowed his protégé sole credit when the article was published in *CMJ*.

In March 1840, when Boole asked his advice on matriculating at Cambridge, Gregory replied that although Boole’s “advanced” age (24) was not necessarily an obstacle, he would have to abandon his original research to follow the prescribed curriculum. Boole decided against attending Cambridge and continued to work as a teacher.

The influence of Gregory’s symbolical algebra on Boole’s mathematics may be seen in the latter’s work. Boole used algebra, freed from the constraints of arithmetic, to establish a formal logic as well as an algebra of sets. The opening sentence of Boole’s first published work on logic, *The Mathematical Analysis of Logic*, reads like a paraphrase of Gregory’s words (see the second quotation in Section 5):

> They who are acquainted with the present state of the theory of Symbolical Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination [Boole 1847, 3].
Boole continued by stating that every system of interpretation to which the same laws of combination apply is equally acceptable, be the system one of numbers, geometry, dynamics, or optics. He claimed:

... the definitive character of a true Calculus, [is] that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation... It is upon the foundation of this general principle, that I propose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its object and in its instruments it must at present stand alone [Boole 1847, 4].

Further echoes of Gregory are heard in An Investigation of the Laws of Thought, in which Boole first defines symbols, and then describes the laws to which they are subject [Boole 1854, 28–29].

4. THE CAMBRIDGE MATHEMATICAL JOURNAL

Archibald Smith (1813–1872) was senior wrangler and first Smith’s prizeman in 1836. While still a Trinity undergraduate, he published a paper in the Transactions of the Cambridge Philosophical Society containing results in solid geometry obtained by use of symmetrical equations; he was one of the first in England to apply such “symmetrical methods” [Dictionary of National Biography 1921, 18:425]. After receiving his degree in 1836, he proposed to Gregory the establishment of “an English periodical for the publication of short papers on mathematical subjects” (quoted in [Smith and Wise 1989, 175]). Gregory deferred doing so until he had completed his own examinations, at which time he founded the Cambridge Mathematical Journal and began publication as its first editor. Although the idea for the CMJ was originally Smith’s, he played a minimal role in the operation of the journal.

While still at Cambridge, Smith was urged to apply for the Glasgow chair of practical astronomy, with the support of Whewell and Airy. Smith was described as “quiet, donnish, and very indecisive” [Smith and Wise 1989, 38]. He explained why he would not apply in a letter to his father in January 1836: “I am tired of Mathematics and look upon a legal life with more pleasure than I used to do” (quoted in [Smith and Wise 1989, 39]). Although he eventually did apply, he lost the position to J. P. Nichol, of King’s College, Aberdeen, who was supported by James and John Stuart Mill.

Although Smith contributed 21 papers to CMJ, Gregory expressed annoyance at what he saw as Smith’s neglect of mathematics. In late 1836, for example, Gregory asked him to comment on a new proof “If you have not now cast away all your mathematics, as you pretended you would do” [Gregory 1836]. In 1839, Gregory complained to Greamhead: “Smith had promised to work, but he has been seized with one of his continual lazy fits and will not put pen to paper. He is going to spend the summer yachting in Scotland, so that I suppose there is no hope of anything from him for a long time” [Gregory 1839].

Late in 1841, it appeared that the natural philosophy chair at Glasgow would soon become vacant, because of the failing health of the occupant Miekleham, and James Thomson began considering applicants. As noted earlier, Gregory was considered for this position, but died before the chair was actually vacated in 1846. The chair went to William Thomson, and it was Smith who was considered to be his most serious competition, despite years spent practicing law rather than mathematics. However, Smith eventually decided he preferred to continue at the bar and later apologized to Thomson for the discomfort caused by his procrastination [Smith and Wise 1989, 114].
In Cambridge, where mathematics occupied such a prominent position, it was to be expected that there would exist a journal devoted to mathematics. Gregory believed that there was a pool of potential contributors at the University and that the publication of their work would encourage others toward original research. In his preface to the first issue of CMJ, he stated that the goal of his journal was “to supply a means of publication of original papers.” He wished to provide a forum for the publication of mathematical papers not so important as to be published in the Transactions of the philosophical societies. A secondary purpose was to publish abstracts of “important and interesting” foreign papers and other works not readily available to the mathematics community at Cambridge. The original articles and the abstracts would cover a variety of subjects, with the emphasis on presenting new ideas, rather than expounding on established ones [Gregory 1837].

After the first issue in November 1837, CMJ continued with publication dates of February, May, and November through Number 24, dated May 1845, after which the journal continued under a different title. Each of the 24 numbers (bound into four volumes) consists of 45 to 50 pages. A trifold page of figures to accompany the papers is inserted at the end of eight numbers. The 269 articles listed in the general index to the four volumes are categorized in that index as dealing with geometry of two dimensions; geometry of three dimensions; algebra and trigonometry; differential calculus and calculus of finite differences; mechanics; astronomy; light and sound; heat, electricity, and magnetism; and miscellaneous. Of the attributed papers, the largest number were contributed by Gregory and by Ellis. Other frequent contributors were Smith, William Walton (who completed Gregory’s Solid Geometry and was editor of Gregory’s collected Mathematical Writings), William Thomson, Greatheed, Arthur Cayley (1821–1895), and Boole. The names Augustus De Morgan (1806–1871), James Joseph Sylvester (1814–1897), George Gabriel Stokes (1819–1903), and William Wallace (1768–1843) appear as well. A number of the papers whose authors are cited in the index had been published anonymously in CMJ. It was Gregory’s policy to allow authors to remain unidentified in order to encourage undergraduates to publish (Ellis, Cayley, and Thomson were among those who did so) without fear for their future reputation.

Gregory remained editor until the fall of 1843 when he became seriously ill and Ellis assumed that role temporarily, until 1845 when William Thomson became editor and renamed the publication the Cambridge and Dublin Mathematical Journal. In Volume I (subtitled Volume V of the Cambridge Mathematical Journal), Thomson included the above mentioned “General Index to the First Series.” In keeping with his newly-established policy of not including anonymous articles, in this index Thomson provided the names (where he was able to determine them) of authors who had originally published under initials in CMJ. Continuing Gregory’s policy, Thomson intended that the publication of new ideas should be the primary focus of the journal. J. J. Sylvester expressed enthusiasm about this goal and offered to submit a series of articles. Later, Thomson decided that it would be worthwhile to include some writings on physical subjects, and it was in the Cambridge and Dublin Mathematical Journal that William Rowan Hamilton published his papers on quaternion geometry.

When Thomson relinquished the editorship in 1852, Norman Macleod Ferrers (1829–1903), a Cambridge mathematician and the editor of George Green’s mathematical papers, became the new editor. In 1857, Ferrers and Sylvester formed a successor journal, the Quarterly Journal of Pure and Applied Mathematics. This publication merged with the Messenger of Mathematics in the 1920’s to form the Quarterly Journal of Mathematics, a journal that is still active.
The *CMJ* was significant for more than its role as the progenitor of a sequence of noteworthy journals. Assessment of the contribution of the publication to mathematics may be seen in contemporaneous as well as later statements. For example, in 1874, Thomson stated that as a new publication, *CMJ* had “inaugurated a most fruitful revival of mathematics in England . . .” (quoted in [Smith and Wise 1989, 176]). De Morgan and Hamilton also praised the quality of the articles in the journal ([De Morgan 1882, 151] and [Graves 1885, 2:528]).

More recently [Smith and Wise 1989, 150], the *CMJ* is described as the center about which the post-1835, second generation mathematical reformers organized themselves, the Analytical Society having been the focus for the first generation. The *CMJ*-centered group was interested in creating and rigorously justifying new methods and emphasized creativity, as opposed to the use of geometry in a rote fashion, as training for the mind.

The writings in the journal, especially on symbolical algebra and its application to the method of separation of symbols, were instrumental in disseminating these ideas. Symmetrical methods were another area that was popularized by papers in *CMJ*. As been already been noted, young men such as Boole and Thomson enhanced their emerging careers with articles published there.

5. GREGORY’S SYMBOLICAL ALGEBRA

Symbolical algebra represented a movement away from algebra as universal arithmetic to a purely formal algebra. It emphasized the importance of structure over meaning, and acknowledged what has been called the principle of mathematical freedom. This principle implies that algebra deals with arbitrary, meaningless symbols, mathematicians create the rules regarding the manipulation of those symbols, and interpretation follows rather than precedes algebraic manipulation. Although Gregory wrote on symbolical algebra in connection with the foundations of algebra in his 1843 paper, “On a Difficulty in the Theory of Algebra,” he was most interested in the subject as it related to the method of separation of symbols of operation from those of quantity [Gregory 1865, p. 1]. The method of separation of symbols, in a nutshell, consisted in classifying the symbols of analysis as either objects (corresponding roughly to the modern notion of function) or operations (corresponding roughly to linear operators), and separating the symbols for the latter type from those of the former type in order to prove theorems in analysis.

A brief examination of the development of symbolical algebra in Britain provides historical background for Gregory’s formulation of symbolical algebra. Those unfamiliar with the specifics of early 19th century mathematical practice in Britain are usually surprised to hear that there was discomfort with the status of negative numbers in many quarters, but a close analysis of the period shows that this concern was an important element in the genesis of British Symbolical Algebra [Allaire 1997]. In Britain prior to 1830, many papers, texts, treatises, and other writings dealt with negative quantities, some accepting them fully. However, there were vocal and influential mathematicians who expressed serious reservations as to the soundness of the foundation of the negatives and therefore of algebra.

Underlying the dissatisfaction was the way mathematics, quantity, and negative were defined at that time. Mathematics was defined as the science of quantity, an Aristotelean category. A quantity was defined as something that could be measured or counted (Aristotle’s magnitude and number), and a negative number was defined as a quantity less than nothing [Barlow 1814, Hutton 1795, Mitchell 1823]. Virtually all writers prior to the early 1800’s
accepted these definitions. Being thus defined, the negatives were manipulated according to rules based on the properties of whole number arithmetic. However, attempts to justify the existence of quantities less than nothing required analogy: loss and gain, forward movement and backward movement, etc. Definition by analogy was the heart of the problem for opponents of the negatives: if a concept could be defined by analogy only, then they would not accept the concept (e.g., [Greenfield 1788]).

There were two alternatives: do algebra without the negatives, or redo algebra with them. There were respected mathematicians of the period, especially Robert Simson (1687–1768), Francis Maseres (1731–1824) and William Frend (1757–1841), who proposed doing algebra without the negatives and were willing to live with the consequences—such consequences as the loss of the degree-roots theorem and much of the theory of equations—while some who were more moderate attempted to find a way to rework algebra. George Peacock (1791–1858) is given credit for being the first to do such a reconstruction in his 1830 Treatise on Algebra [Peacock 1830], although documents have been found indicating that he may have drawn from earlier writings of Babbage and Woodhouse [Dubbey 1977, Becher 1980].

Peacock developed symbolical algebra by beginning with the algebra of non-negative reals, which he called Arithmetical Algebra, algebra with all the restrictions of arithmetic. General symbols are restricted in the values they may assume, so that zero is an absolute minimum, the expressions $-a$ and $\sqrt{-a}$ are meaningless, as is $a - b$ when $a$ is less than $b$. Polynomial equations would have only non-negative real roots. Then he constructs Symbolical Algebra, which he defines as

*the science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws: for we may assume any laws . . . , so long as our assumptions are independent, and therefore not inconsistent with each other . . . [Peacock 1830, 71].*

Thus symbols are to represent arbitrary quantities, unlimited in their nature and their magnitude. It follows that $a - b$ is always possible and $-c$ is allowed. Peacock almost has formal algebra, but falters at the final step. Believing that algebra will be pointless unless the symbols are interpreted, he requires that the operations of arithmetic should be those chosen for algebra. He extends the laws of arithmetic to algebra by postulating the principle of permanence of equivalent forms, which states that if there is an equivalent form in arithmetical algebra, e.g., $a^n \times a^m = a^{n+m}$, such must be the form in symbolical algebra. Thus $a^n \times a^m$ is $a^{n+m}$ in any interpretation. This principle is, as Gregory will argue, the weak link in Peacock’s symbolical algebra.

Gregory, as well as William Rowan Hamilton (1805–1865) and De Morgan, were among the next generation of symbolical algebraists, who examined, criticized, and improved on Peacock’s work. For the remainder of this section, we undertake a close analysis of Gregory’s paper “On the Real Nature of Symbolical Algebra” [Gregory 1840], as reprinted in [Gregory 1865]. At first glance, Gregory’s definition of symbolical algebra does not appear to be noticeably different from Peacock’s:

... symbolical algebra is ... the science which treats of the combination of operations defined not by their nature, ... but by the laws of combination to which they are subject, ... [W]e suppose the existence of classes of unknown operations subject to the same laws [Gregory 1865, 2].

However his refinements are significant. Gregory adopted Peacock’s approach, which he attempted to clarify or modify where he thought necessary. He isolates what he considers to
be the most primitive relations that must exist between operations in order for the various proofs to be valid. These relations are independent of any interpretation of the symbols. He assumes five classes of operations, and defines operations as belonging to a class if they are “subject to the same laws,” not “by what they are or what they do.” What follows is a description of Gregory’s classes, using his notation [Gregory 1865, 3–13].

Note that in what follows, \( f \) and \( F \) should not be read as function notation; rather, they are the operations, performed on the symbols to which they are prepended. The objects \( a, b, x, \) and \( y \) are usually functions, but also sometimes quantities, geometrical figures or even other operators, such as the differential operator as used in class IV. Thus, Gregory views an operation as a fundamental and very general concept. We observe, although Gregory did not, that all of these operations are unitary, and all map objects of a certain type to objects of the same type. Gregory makes additional implicit assumptions about the objects, such as the fact that there is a binary operation corresponding to addition defined on them (e.g., classes III, IV and V).

I. The two classes of circulating operations, \( F \) and \( f \), connected by the laws

\[
\begin{align*}
(1) \quad FF(a) &= F(a) \\
(2) \quad fF(a) &= f(a) \\
(3) \quad Ff(a) &= f(a) \\
(4) \quad ff(a) &= F(a)
\end{align*}
\]

An obvious interpretation of \( f \) and \( F \) is as \(+\) and \( -\) with the laws giving the usual sign rules for multiplication. But Gregory notes that other interpretations are possible, e.g., \( f \) as rotation through \( \pi \) and \( F \) as rotation through \( 2\pi \).

II. The class of index operations, satisfying the laws

\[
\begin{align*}
(1) \quad f_m(a) \cdot f_n(a) &= f_{m+n}(a) \\
(2) \quad f_m f_n(a) &= f_{mn}(a)
\end{align*}
\]

where \( f_m \) and \( f_n \) are “different species of same genus of operations.” Gregory offers interpretations for when \( m \) and \( n \) are integers or rational number, and says that we may suppose them to “indicate any operation whatever.” However, he confesses “I do not know of any interpretation which can be given to the notation, excepting in the case when it indicates the operation of differentiation, represented by the symbol \( d \);” referring to the operator form of Taylor’s theorem:

\[
e^{h \frac{d}{dx}} f(x) = f(x + h).
\]

For Peacock, using the principle of permanence of equivalent forms, these general index laws would follow from their validity when considered as laws of arithmetic involving integers. On the other hand, Gregory would say that to confirm the validity of \( f_m(a) \cdot f_n(a) = f_{m+n}(a) \) for nonintegers, one would have to prove it directly from the nature of the given operation and the given subjects.

III. The class of distributive and commutative operations. The first use of these names is generally credited to François-Joseph Servois (1768–1847) [Servois 1814a]. This class
is subject to the laws

\begin{align*}
(1) \quad f(a) + f(b) &= f(a + b) \\
(2) \quad f_1f(a) &= ff_1(a).
\end{align*}

Operations in this class are differentiation, integration, and the taking of finite differences. A geometric interpretation is the operation of “transference to a distance measured in a straight line.” Although Gregory defines neither − nor +, if the former is taken to mean is the same figure as and the latter to mean union, the laws hold.

IV. The unnamed class of operations defined by

\[ f(x) + f(y) = f(xy). \]

\( f \) can be seen as the operation of the arithmetical logarithm when \( x \) and \( y \) are numbers. General theorems such as

\[ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \]

are proved using only the “laws we have laid down” and will therefore apply to any other operations subject to the law. Gregory calls \( \log(1 + x) \) an abbreviation for the series on the right. He says that he knows of no interpretation when \( x \) is not a number, but conjectures that

\[ \log \left(1 + \frac{d}{dx}\right) \]

may have meaning.

V. The final class, also unnamed, involving two operations related by the conditions

\begin{align*}
(1) \quad aF(x + y) &= F(x)f(y) + f(x)F(y) \\
(2) \quad af(x + y) &= f(x)f(y) - cF(x)F(y).
\end{align*}

If \( a \) and \( c \) are unity, (1) and (2) are the familiar forms for the combinations \( \sin(x + y) \) and \( \cos(x + y) \), functions of circular sectors. If \( a \) and \( c \) are not unity, Gregory states that the forms are “suggested by the known relations between certain functions of elliptical sectors.” In this case, \( c \) depends on both the length \( a \) of the semi-major axis and the length of the semi-minor axis of the ellipse in question; see [Allaire 1997, 96–98].

He remarks that in algebra the forms \( \sin x \) and \( \cos x \) are abbreviated forms for the Taylor series relations among the commutative and distributive, circulating, and index operations. He cites as the most important theorem proved of this class of operations, DeMoivre’s theorem, which he writes as

\[ \left\{ \cos x + (-)^{\frac{1}{2}} \sin x \right\}^n = \cos nx + (-)^{\frac{1}{2}} \sin nx, \]

where \( (-)^{\frac{1}{2}} \) is Gregory’s notation for the imaginary \( i \), underscoring his view that \( i \) is an operation, akin to \( - \) (negation), rather than a quantity, akin to \( -1 \).
The key points of symbolical algebra as refined by Gregory are:

1. What can be proved for a class generally, holds for all specific operations in that class. (Although this may seem tautologous to us, Gregory clearly felt this point needed to be stated explicitly.)

2. A theorem, e.g., the binomial theorem involving classes I, II, and III, is a relation between different classes of operations, expressed in symbols. Further, “if we can show that any operations in any science [i.e., branch of mathematics] are subject to the same laws of combination as these classes, the theorems are true of these as included in the general case: Provided always, that the resulting combinations are all possible in the particular operation under consideration.”

3. If the combination of two operations in a science is not possible in that science, then the previous statement cannot hold, e.g., $\sqrt{-1}$ in arithmetic.

4. A theorem carries from one science to another not because of any analogy between the operations nor any similarity in the operations, but because the operations involved are subject to the same laws of combination. Gregory is explicitly contrasting his position to that of Lagrange, who spoke in [Lagrange 1772] of the analogy between the binomial theorem and Leibniz’s differentiation formula (Theorem XV.7 in Section 7).

5. Arithmetic may be the suggesting science for many of the laws of combination, but such is not necessarily the case. In class I, for example, the circulating operations are also suggested by rotations.

Gregory disagrees with Peacock’s principle of the permanence of equivalent forms, and offers items 2 and 3 as an alternative. In item 5, Gregory goes beyond Peacock and breaks the ties to arithmetic.

The object of this paper, which Gregory considered to be his manifesto for an algebraic approach to analysis, was to refine the principles of symbolical algebra for the purpose of placing the method of separation of symbols on a “firm and secure basis.” Gregory believed that this method was a powerful one, which had the potential for wide application in algebra and in many other fields.

It is from the laws I, II, and III that Gregory justified his technique of separation of symbols of operation from those of quantity. In this method, symbols for the operations are distinguished from symbols of quantity, and the former are treated as if they were “ordinary algebraic symbols” [Gregory 1865, 7].

As an example, consider the derivative $\frac{dy}{dx}$; taking the derivative is the “operation,” $y$ is the “quantity.” Hence, separating symbols of operation from symbols of quantity, the expression $2\frac{dy}{dx} + y$

may be written as

$$\left(2\frac{d}{dx} + 1\right)y$$

or even $(2D + 1)y$ where the variable of differentiation is understood. The method, with iteration expressed by means of an exponent, is used in the differential operator technique
for solving linear differential equations. This will be discussed in greater detail in Section 7. Differential operators, of course, are distributive in the sense of Gregory’s rule III.1, and commutative in the sense of III.2, with \( f_i \) representing the operation of multiplication by a constant coefficient.

Gregory viewed the binomial theorem as the most important in symbolical algebra because it expresses a relationship among the circulating, index, and commutative and distributive operations. The binomial theorem, he asserts, holds for all operations that are commutative and distributive and, therefore Euler’s demonstration of the binomial series expansion for fractional and negative powers is correct. In fact, Gregory continues, the binomial theorem applies to such forms as

\[
(1 + a)^{\frac{d}{dx}}
\]

because \( a \) and \( \frac{d}{dx} \) are distributive and commutative operations. It does not apply, however, to

\[
(1 + f(x))^{\frac{d}{dx}}
\]

because \( f(x) \) and \( \frac{d}{dx} \) do not commute with one another. Nor does it apply to \( (1 + a)^{\log} \), because \( \log \) is not of the same class as the other quantities. This last observation he believes to be uniquely his [Gregory 1865, 9–10].

Gregory applied his method of separation of symbols to a wide variety of problems, and published the results in various papers in the CMJ. Problems include the solution of linear differential equations with constant coefficients [Gregory 1865, p. 14–27], linear equations of finite and mixed differences [33–42], partial differential equations [62–72], functional equations [247–256], the integration of simultaneous differential equations [95–106], demonstrations of theorems in the differential calculus and calculus of finite differences [108–123], consideration of the sympathy of pendulums [175–183], the evaluation of definite multiple integrals [204–215], the relationship between factorials and binomial expansions [226–229].

6. THE EXAMPLES

*Examples of the Processes of the Differential and Integral Calculus* was published in 1841. The object of this work was “to furnish to the student examples by which to illustrate the process of the Differential and Integral Calculus.” [Gregory 1841a, p. iii]. Peacock had published a similar collection, *Collection of Examples of the Applications of the Differential and Integral Calculus*, in 1820 [Peacock 1820]. Peacock said that his work was originally intended as a supplement to his 1816 translation of Lacroix’s calculus. In the course of writing his book, however, Gregory decided to expand it to cover applications of all important ideas in a manner free of reference to any specific work since virtually all presented theory without illustrations or examples [Gregory 1841a, iii–iv].

By about 1840, Peacock’s book of examples was out of print and, according to De Morgan [De Morgan 1842, p. viii], difficult to obtain. When Peacock declined to create a second and/or revised edition, Gregory, believing that such a work was of value to the students of Cambridge, prepared his own work, *Examples of the Processes of the Differential and Integral Calculus*. (Subsequent references to this text will be simply as *Examples*.)
Like the title, the purpose of the work was similar, but not identical, to Peacock’s. Gregory intended to provide not only a collection of illustrative problems and examples but also demonstrations of what he described as important and interesting propositions not usually found in calculus texts of the time. Reflected in his revision were “recent improvements of the calculus and . . . the new requirements of natural philosophy [that] have greatly influenced the progress of pure analysis” [Ellis 1845, 149]. The book was to be complementary to all “elementary treatises.” He used only methods employed in such treatises, with one exception; he used the Method of Separation of Symbols since he believed that this method, which some other writers did not accept, shortened and simplified many of the processes.

In this volume, Gregory cites the source for many examples, in particular “all cases in which the student would be likely to wish for more information by consulting original authors [Gregory 1841a, vi].” Another interesting feature of the text is that he provides historical notes where he deems it appropriate:

> It has always appeared to me that we sacrifice many of the advantages and more of the pleasures of studying any science by omitting all reference to the history of its progress: I have therefore occasionally introduced historical notices of those problems which are interesting either from the nature of the questions involved, or from their bearing on the history of the Calculus. . . . [T]hese digressions may serve to relieve the dryness of a mere collection of Examples [Gregory 1841a, vi].

An example of such a “notice” is the one given in connection with problems dealing with the cycloid. He presents a two-page history of that curve beginning in the mid-15th century, discussing the origin of the cycloid and contributions of Galileo, Descartes, Pascal, Wallis, Huygens, and John Bernoulli. He cites it as the solution to the brachistochrone problem, and provides references to several treatises on the calculus.

In the differential calculus section, he presents problems on the usual topics such as differentiation, indeterminate functions, extrema, tangents, normals and asymptotes, curve tracing, and applications to solid geometry. In Chap. XV, the concluding chapter to Part I, Gregory provides a brief exposition of the theory of separation of symbols. The remainder of the chapter is a collection of the theorems of differential calculus that may be proved by this method. (See Section 7 for a more complete description of the contents of this chapter.)

The chapters in Part II, Integral Calculus, deal with such topics as the various techniques of integration, differential equations, partial differential equations, areas and volumes, geometric problems involving differential equations, and definite integrals. The first three chapters are devoted to techniques of integration. Beginning with Chapter IV, “Solution of Differential Equations,” Gregory begins to use separation of symbols.

Ball, describing the mathematics text books at Cambridge, remarks that Examples “was a work of great ability and was one of the earliest attempts to bring the calculus of operations into common use” [Ball 1889, 130].

De Morgan, in the preface to his 1842 Differential and Integral Calculus, “strongly recommend[s]” Gregory’s Examples, describing it as “containing instances of all the latest and best modes of treating the details of the differential and integral calculus” (viii).

7. USES OF SYMBOLICAL ALGEBRA IN THE EXAMPLES

The capstone of Part 1 of the Examples is its final chapter: “General Theorems of the Differential Calculus.” Indeed, this must be seen as the cornerstone of the entire text. For one thing, it represents the most significant departure from Peacock’s earlier version of the
Examples. Furthermore, since it is a collection of theorems, most of which had recently been discovered or given novel proofs, it is the most current portion of the Examples, and the part that most clearly reflects contemporary trends in mathematics at Cambridge. Additionally, it establishes techniques that will be used extensively in Part II, in the solution of differential equations. Finally, as an exposition of the method of separation of symbols, it most clearly reflects Gregory’s agenda for algebraic methods in analysis. If the 1838 essay “On the Real Nature of Symbolical Algebra” [Gregory 1840] is Gregory’s manifesto for the proper approach to analysis, then Chapter XV of the Examples is the progress report, just a few years later, on the successes of that approach.

“In this chapter,” says Gregory, “I shall collect those Theorems in the Differential Calculus which, depending only on the laws of combination of the symbols of differentiation, and not on the functions which are operated on by these symbols, may be proved by the method of separation of symbols.” Since the method has “not yet found a place in the elementary works on the Calculus,” Gregory begins with an exposition of the axioms and techniques of symbolical calculus. Although the exposition is his own, Gregory acknowledges his debt to Servois, by whom “the method of separation of symbols was first correctly given” (in [Servois 1814a]), and to Robert Murphy (1806–1843), who had recently added “some very valuable researches on the subject” (in [Murphy 1837]). [Gregory 1841a, 233–235]

Gregory’s exposition here amounts to a streamlined version of his 1838 essay on separation of symbols [Gregory 1840], discussed in Section 5. He concentrates on points II and III of that earlier system, and uses notation that is less confusing than the functional notation of the earlier exposition.

Let \( u \) and \( v \) represent functions. Then we will consider operations \( a \) and \( b \) on these functions which satisfy the following three axioms:

\[
ab(u) = ba(u) \tag{1}
\]
\[
a(u + v) = a(u) + a(v) \tag{2}
\]
\[
am anu = am a^n u. \tag{3}
\]

As in Section III of [Gregory 1840], Gregory calls the first of these the **commutative law** and the second the **distributive law**. “The third law,” says Gregory, “is not so much a law of combination of the operation denoted by \( a \), but rather of the operation performed on \( a \), which is indicated by the index . . . It may be conveniently called the law of repetition.” The most important examples of the law of repetition occur when \( m \) and \( n \) are positive integers representing the number of times the operation \( a \) is repeatedly applied to the function \( u \). However, Gregory stresses the purely formal status of the indices in this law and explicitly allows for the possibility of negative and non-integer indices [Gregory 1841a, 233–234].

As an example of a model for this system, Gregory gives

\[
a = \frac{d}{dx}
\]
\[
b = \frac{d}{dy}.
\]

In view of the commutative law, a modern author would take care to restrict the functions
Gregory remarks that, in this model, the obvious interpretation of negative indices in the law of repetition is the indefinite integral, and further remarks “I shall not however make any use of the interpretation of the formulæ when the indices of differentiation are fractional” [Gregory 1841a, 235].

In this interpretation, which is the intended one for many proofs in analysis, the operations $a$ and $b$ might also represent multiplication by a constant. In stating the axioms and mentioning this interpretation, Gregory has in a few short paragraphs summarized the work done by Servois in [Servois 1814a], who demonstrates that any class of operations which are distributive and commutative, commutative both amongst themselves and with the operation of multiplication by a constant, is closed under sums and composition (Servois’ results will be discussed in greater detail in Section 8).

Any algebraic theorem that depends only on these three laws will have an analytic analog. As in [Gregory 1840], Gregory emphasizes among these the central position of the binomial theorem. The distributive and commutative laws may be used repeatedly in order to express $(x + y)^n$ as a sum of $2^n$ terms. Those terms can then all be put into the form $x^i y^j$ by repeated application of the commutative law and the law of repetition. Finally, like terms are gathered, once again using the distributive law and the law of repetition. We note that in this example, whose details Gregory left to the reader, the operations $a$ and $b$ are variously multiplication by or addition to various terms involving $x$ and $y$. Gregory never states separate commutative laws of addition and of multiplication; instead he uses this more general notion of commutativity, with the operations being multiplication by a given term or addition to a given term. Similarly, the combined power of the law of repetition and the commutative law eliminates the need for the associative laws, which one would see in a more modern treatment.

Gregory proves 18 propositions in this chapter, to which we will refer as (XV.1) through (XV.18). The first of these general results, (XV.1) is simply a recasting of Taylor’s theorem in operator form. “This theorem,” says Gregory, “may be reduced into a very convenient shape by the separation of symbols.” Since,

$$f(x + h) = f(x) + \frac{h}{1} \frac{d}{dx} f(x) + \frac{h^2}{1 \cdot 2} \left( \frac{d}{dx} \right)^2 f(x) + \frac{h^3}{1 \cdot 2 \cdot 3} \left( \frac{d}{dx} \right)^3 f(x) + \cdots,$$

then separating symbols, Gregory has

$$f(x + h) = \left[ 1 + h \frac{d}{dx} + \frac{h^2}{1 \cdot 2} \left( \frac{d}{dx} \right)^2 + \frac{h^3}{1 \cdot 2 \cdot 3} \left( \frac{d}{dx} \right)^3 + \cdots \right] f(x).$$

Gregory observes that the quantity in the square braces is the series expansion for the quantity $e^{hx}$, with $x$ replaced everywhere by $\frac{d}{dx}$. Then, “as the symbol $\frac{d}{dx}$ is subject to the same laws of combination as the symbol $x$ is supposed to be subject in the demonstration of the exponential Theorem,” he expresses Taylor’s theorem in the compact notation

$$f(x + h) = e^{\frac{d}{dx}} f(x).$$
This leads to the definition of the operation $E_h$ by the formula

$$E_h f(x) = f(x + h)$$

and yet another alternate expression for Taylor’s theorem:

$$E_h f(x) = e^{h \frac{d}{dx}} f(x).$$

Finally, since

$$E_h E_k f(x) = E_h f(x + k) = f(x + h + k) = E_{h+k} f(x),$$

the index law applies to the family of operations $E_h$, and this is a concrete example of noninteger values in the law of repetition [Gregory 1841a, 235–236].

Although Gregory does not mention the fact, the family of operations $E_h$ is commutative and distributive in addition to satisfying the law of repetition, and so provides an entirely different class of operations satisfying Gregory’s 3 axioms, the translation or finite difference operators. Gregory credits Servois with the use of the symbol $E$ for the translation operation. Servois, who calls the function the varied state ($l’´etat vari´e$), credits Arbogast with establishing this notation [Servois 1814a, 94] in the text Du calcul des d´erivations [Arbogast 1800].

The evolution of this form of Taylor’s Theorem actually begins with Joseph Louis Lagrange (1736–1813), who observed in 1772 that

$$\Delta u = e^{h \frac{d}{dx}} u - 1,$$

where $\Delta u = u(x + h) - x(x)$ (or $Eu(x) - u(x)$ in Arbogast’s notation). It was Arbogast who then cast this in the language of operators, separating symbols of operation from symbols of quantity, as Gregory would say. In particular, he gave Lagrange’s equation the form

$$\Delta u = (e^{h \frac{d}{dx}} - 1)u,$$

whence

$$E = 1 + \Delta = e^{h \frac{d}{dx}}.$$

For more details on this evolution from Lagrange, whose “formulas do not have the character of operational identities,” but rather “the character of mechanical rules,” to the algebraic treatment of Servois, see [Lusternik and Petrova 1972].

Given the status of the binomial theorem as the sine qua non of symbolical algebra, it’s not surprising that Gregory’s second result in this chapter, (XV.2), is a binomial-type theorem: a derivation of the $n$th order total differential for a function of two variables. If $u$
is a function of \( x \) and \( y \), then

\[
d(u) = \frac{du}{dx} dx + \frac{du}{dy} dy
\]

\[= \left( \frac{d}{dx} dx + \frac{d}{dy} dy \right) u,
\]

where the second line follows from the first by “separating the symbol of operation from the subject.” Gregory then affixes “the general symbol \( n \) as an index to the operations on both sides of the equation,”

\[
d^n(u) = \left( \frac{d}{dx} dx + \frac{d}{dy} dy \right)^n u,
\]

and simply expands “the operation on the second side by the Binomial Theorem.” The result is

\[
d^n(u) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^n u}{dx^{n-k} dy^k} dx^{n-k} dy^k.
\]

Gregory’s expression is as follows:

\[
\frac{d^n u}{dx^n} dx^n + n \frac{d^n u}{d(x^{n-1}) dy} d(x^{n-1}) dy + \frac{n(n-1)}{1 \cdot 2} \frac{d^2 u}{dx^{n-2} dy^2} dx^{n-2} dy^2 + \text{&c.}
\]

This result is followed by (XV.3), a multinomial theorem for \( d^n(u) \) when \( u \) is a function of three or more variables [Gregory 1841a, 236–237].

A few pages later Gregory derives (XV.7), Leibniz’ formula:

\[
\left( \frac{d}{dx} \right)^n (uv) = v \frac{d^n u}{dx^n} + n \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2} \frac{d^2 v}{dx^2} \frac{d^{n-2} u}{dx^{n-2}} + \text{&c.}
\]

Gregory notes that although Leibniz’ original argument was an induction on positive integers, the result “...is true whether \( n \) be integer or fractional, positive or negative.” As with the previous binomial-like theorem, Leibniz’ theorem is immediately generalized in (XV.8) to a product of three or more functions. [Gregory 1841a, 239]

The next of the general theorems, (XV.9), is the case of a negative index in Leibniz’ theorem, yielding a general form for integration by parts. In (XV.10), the case \( u = 1 \) in Theorem (XV.9) is considered. Specifically, Bernoulli’s series

\[
\int dx (v) = xv - \frac{x^2}{1 \cdot 2} \frac{dv}{dx} + \frac{x^3}{1 \cdot 2 \cdot 3} \frac{d^2 v}{dx^2} - \text{&c.}
\]

is derived. Note that in Gregory’s system \( \int dx \) is the symbol of an operation, and it is applied to the symbol \( v \). Theorem (XV.11) is the case \( v = e^{ax} \) in Leibniz’ theorem, which we will consider in detail in the next section.
Chapter XV concludes with a sequence of results concerning finite differences, and the connections between these and the differential operators. Included are results originally due to Siméon Denis Poisson (1781–1840) (XV.13), Murphy (XV.14), and François Joseph Français (1768–1810) (XV.16).

8. SYMBOLICAL ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

In the previous section, we skipped over Theorems (XV.4) through (XV.6), which are geared to the solution of differential equations.

Suppose the polynomial

\[ p(x) = x^n + A_1 x^{n-1} + A_2 x^{n-2} + \cdots + A_{n-1} x + A_n \]

may be factored as

\[ (x - a_1)(x - a_2)\cdots(x - a_n), \]

where no assumption is yet made concerning whether the roots \( a_1, a_2, \ldots, a_n \) are real or complex, distinct or repeated. It follows, using the laws of separation of symbols, that the differential operator

\[ \frac{d^n u}{dx^n} + A_1 \frac{d^{n-1} u}{dx^{n-1}} + A_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + A_n \frac{d^n u}{dy^n} \]

is equal to

\[ \left( \frac{d}{dx} - a_1 \frac{d}{dy} \right) \left( \frac{d}{dx} - a_2 \frac{d}{dy} \right) \cdots \left( \frac{d}{dx} - a_n \frac{d}{dy} \right) u. \]

This result, (XV.4), is specialized in (XV.5) when \( u \) is a function of \( x \) only, as follows. If the constants \( A_i \) and \( a_i \) are as above, then

\[ p \left( \frac{d}{dx} \right) u = \frac{d^n u}{dx^n} + A_1 \frac{d^{n-1} u}{dx^{n-1}} + A_2 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + A_n u \]

is equal to

\[ \left( \frac{d}{dx} - a_1 \right) \left( \frac{d}{dx} - a_2 \right) \cdots \left( \frac{d}{dx} - a_n \right) u. \]

This result is often employed in the solution of ODEs with constant coefficients in contemporary textbooks. When applied to the equation,

\[ \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2 y = 0, \]
for example, we have, by separating symbols, the auxiliary equation

$$(D^2 + 3D + 2)y = 0.$$  

We are denoting the operator $\frac{d}{dx}$ as $D$ since there is no confusion over the variable of differentiation. By (XV.5) the expression in parentheses may be factored as though it were an ordinary polynomial:

$$(D + 1)(D + 2)y = 0.$$  

The particular solutions $y = e^{-x}$ and $y = e^{-2x}$ of the equation are then recognized to be the solutions to $(D + 1)y = 0$ and $(D + 2)y = 0$.

Theorem (XV.6) provides Gregory’s justification for passing from the solutions of the simple equations $(D - a_i)y = 0$ to the general solution as a linear combination of such particular ones.

Suppose the roots $a_1, a_2, \ldots, a_n$ of the polynomial equation $p(x) = 0$ are all distinct. Then rational function decomposition yields

$$\frac{1}{x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n} = \frac{N_1}{x - a_1} + \frac{N_2}{x - a_2} + \cdots + \frac{N_n}{x - a_n},$$

where

$$N_i = \frac{1}{\prod_{k \neq i}(a_i - a_k)}$$

for $i = 1, 2, \ldots, n$.

Therefore,

$$\frac{1}{p\left(\frac{d}{dx}\right)} u = \left[\frac{d^n}{dx^n} + A_1\frac{d^{n-1}}{dx^{n-1}} + A_2\frac{d^{n-2}}{dx^{n-2}} + \cdots + A_n\right]^{-1} u$$

is equal to

$$\sum_{i=1}^{n} N_i \left(\frac{d}{dx} - a_i\right)^{-1} u,$$

with an analogous result when $u$ is a function of $x$ and $y$, as in Theorem (XV.4). Furthermore, if a root $a$ has multiplicity $r > 1$, then the decomposition of the operator

$$\left[\frac{d^n}{dx^n} + A_1\frac{d^{n-1}}{dx^{n-1}} + A_2\frac{d^{n-2}}{dx^{n-2}} + \cdots + A_n\right]^{-1}$$
contains the series of \( r \) terms

\[
\sum_{p=1}^r M_p \left( \frac{d}{dx} - a \right)^{-p},
\]

where \( M_p \) is the value corresponding to \( x = a \) in the expression

\[
\frac{1}{(r - p)!} d^{r-p} \frac{1}{dx^{r-p} q(x)},
\]

where the polynomial \( q(x) \) is defined by \( p(x) = (x - a)^{r} q(x) \).

At first glance, Theorem (XV.6) seems both technical and of little importance. Gregory credits the result to George Boole, from an article in Volume II of the CMJ [Boole 1840] which, as we have previously discussed, was in fact written up and diagrammed by Gregory himself. Lest we be inclined to dismiss its inclusion here as nothing but Gregory’s promotion of his young protégé, we should examine how ordinary differential equations with constant coefficients were solved by Gregory in Chapter IV of Section II of the Examples. The method Gregory exposes was widely used in the 19th century [Lusternik and Petrova 1977].

Consider the general form of a linear differential equation with constant coefficients,

\[
y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = f(x),
\]

where \( y \) is a function of \( x \) only. Then

\[
p(D)y = f(x)
\]

where, once again, we use \( D \) to represent the operator \( \frac{d}{dx} \), and the polynomial \( p(x) \) is as above. Formally dividing through by \( p(D) \), we have

\[
y = \frac{1}{p(D)} f(x).
\]

Thus if the roots of \( p(x) = 0 \) are all real and distinct, we have by (XV.6)

\[
y = \left[ \sum_{i=0}^n \frac{N_i}{D - a_i} \right] f(x).
\]

The right hand side may now be expanded to yield the general solution as a sum of particular solutions,

\[
y = \sum_{i=0}^n y_i,
\]

where

\[
y_i = \frac{N_i}{D - a_i} f(x).
\]
A formal multiplication yields

\[(D - a_i)y_i = N_i f(x)\].

The general solution is completed by solving these simple first order linear equations and summing the terms.

A modern treatment would first consider the corresponding homogeneous equation

\[(D - a_i)y = 0\].

Its solution, which Gregory calls the \textit{complementary function} is \(c_i e^{a_i x}\), where \(c_i\) is an arbitrary constant. A particular solution to the non-homogeneous equation would be given as follows:

\[N_i \int_0^x e^{a_i(x-t)} f(t) \, dt.\]

The general solution to (5) would then be the sum of the homogeneous solution (with its arbitrary constant) and the particular one. Having determined that the general solution is the sum of such \(y_i\)'s, we have

\[y = \sum_{i=0}^n c_i e^{a_i x} + \sum_{i=0}^n N_i \int_0^x e^{a_i(x-t)} f(t) \, dt.\] (6)

Gregory’s solution of the differential equation (5) is justified by his general Theorem (XV.11), which we will now derive. In Leibniz’ theorem (XV.7), let \(v = e^{ax}\). Then, as

\[\frac{d^k v}{dx^k} = a^k e^{ax} = a^k v,\]

we have

\[\left( \frac{d}{dx} \right)^n u e^{ax} = e^{ax} \left[ \left( a + \frac{d}{dx} \right)^n u \right].\]

Dividing through by \(e^{ax}\) yields Theorem (XV.11):

\[\left( a + \frac{d}{dx} \right)^n u = e^{-ax} \left( \frac{d}{dx} \right)^n e^{ax} u.\]

To solve Eq. (5), we specialize Theorem (XV.11) with \(n = -1\), \(a = -a_i\) and \(u = f(x)\). In Chap. IV of Part II of the \textit{Examples}, Gregory observes that

\[\left( \frac{d}{dx} - a_i \right)^{-1} f(x) = e^{a_i x} \left( \frac{d}{dx} \right)^{-1} e^{-a_i x} f(x) = e^{a_i x} \int dx \, e^{-a_i x} f(x).\]

Furthermore, “it is to be observed that each of the signs of integration would give rise to an arbitrary constant; and that this must be added in each of the terms when the integration is
affected.” The general solution is therefore

\[ y = \sum_{i=1}^{n} N_i e^{a_i x} \left( \int dx e^{-a_i x} f(x) + C_i \right). \]

To formulate this expression in modern terms, we must replace the ambiguous integral with the following

\[ \int_{0}^{x} e^{-a_i t} f(t) \, dt. \]

Bringing the terms \( e^{a_i x} \) inside the integral and replacing the constants \( N_i C_i \) by \( c_i \) reconciles Gregory’s solution with Eq. (6) [Gregory 1841a, p. 288].

In the case of repeated roots, the equation (4) involves terms of the form

\[ \left( \frac{d}{dx} - a \right)^{-p}, \]

which corresponds to the case \( n < -1 \) in Theorem (XV.11). Thus, the solution involves multiple integration, and hence arbitrary constants applied to all powers of \( x \) from 0 through \( r - 1 \), where \( r \) is the multiplicity of the root. After a general discussion, Gregory gives the explicit form of the solution in the case where the roots \( a_1 \) through \( a_{n-r} \) are distinct, and there is a root \( a \) of multiplicity \( r \), as a concrete example [Gregory 1841a, p. 289].

Gregory next considers the case of a conjugate pair of complex roots, or “impossible” roots as he refers to them

\[ \alpha + (\frac{-1}{2}) \beta \quad \text{and} \quad \alpha - (\frac{-1}{2}) \beta. \]

As one might expect, the solution is given in terms of sines and cosines instead of complex numbers, just as a modern text would do. This completes the discussion, for “if there be a number of equal pairs of impossible roots in the equation, the general expression for the value of \( y \) becomes so complicated as to be of little use, and it is therefore unnecessary to insert it here” [Gregory 1841a, 290].

9. SERVOIS, MURPHY, AND THE EVOLUTION OF SYMBOLIC CALCULUS

Symbolical algebra has also been called the algebra of operations, the symbolic calculus, the functional calculus, the calculus of operations, and various other similar names. It originated with Lagrange, and although it did not lead to a satisfactory foundation for the differential and integral calculus, it both inspired the development of abstract algebra in 19th century Britain, and led directly to linear operator theory in the 20th century. Koppelman [Koppelman 1971] identifies a 1695 letter from Leibniz to Johann Bernoulli, describing the analogy between the binomial theorem and the general product rule for differentials (Gregory’s theorem (XV.7)), as the first step in the development of the symbolical calculus. This is followed by two great periods of activity: one in France beginning with a 1772 paper by Lagrange [Lagrange 1772] and culminating in the work of Augustin Louis Cauchy
Gregory specifically references Lagrange’s 1772 paper in the early pages of the Examples, although he does not discuss it in the crucial Chap. XV. Between this paper of Lagrange and the work of Servois which we are about to discuss, other French mathematicians—Arbogast, Brisson, Français, and Lacroix—made important contributions to the symbolic calculus. However, these are not discussed in the Examples.

Gregory’s place in this development is largely as an expositor and popularizer. The algebraic method of solving ODEs, for example, which we described in the previous section, is largely due to Barnabé Brisson (1777–1828), who read two papers on the method to the Paris Academy in 1821 and 1823. Although these papers were never published, they were lauded, and expanded upon, by Cauchy in a series of three papers in 1827. For his own part, Gregory’s original contribution to the method is Theorem (XV.11), proved in a paper early in the first volume of the CMJ [Gregory 1865, 14–27]. His lasting contribution, though, was to bring this together with Boole’s theorem (XV.6) and the work of Brisson and Cauchy in the Examples to make a complete and coherent method available to Cambridge undergraduates and, indeed, the entire English speaking mathematical community.

In this section we will briefly examine the work of a practitioner in each of the two schools of the symbolic calculus: Servois in France and Murphy in England. As we have seen, these two were singled out for special mention by Gregory in Chap. XV of Part I of the Examples.

François-Joseph Servois was born to an itinerant merchant in the French Alps in 1768. Over the course of his life, he had four careers: priest, soldier, mathematician, and museum curator. Like so many young clerics in France at the time of the French Revolution, he left the priesthood in the early 1790s, in this case to be an artillery officer. After seven years of active and distinguished service, he was assigned to professorial duties, on the advice of Legendre. From 1801 through 1816 he served as an instructor or professor of mathematics at various French military schools and established his reputation as a mathematician. Most of his publications appeared in Gergonne’s Annales des mathématiques pures et appliquées between 1811 and 1814. He was a staunch believer in the efficacy of algebraic formalism in mathematics, and his reputation in projective geometry was such that his advice was repeatedly sought by Poncelet in the preparation of the latter’s 1822 Traité des propriétés projective.

In 1814 he published two papers on the foundations of analysis in the Annales [Servois 1814a, Servois 1814b], which he also arranged to have privately published as a single 80-page monograph. The first of these, “Essai sur un nouveau mode d’exposition des principes du calcul différentiel,” was a work of pure mathematics concerning the symbolic calculus, while the second, “Reflexions sur les divers systèmes d’expositions des principes du calcul différentiel, et, en particulier, sur la doctrine des infiniments petits,” was essentially a work in the philosophy of mathematics, including a historical survey of attempts to provide a foundation for the calculus; a defense of Lagrange’s algebraic doctrine against a recent attack by Josef Hoëné de Wronski (1778–1853); and a blistering condemnation of the doctrine of infinitesimals.

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4 The biographical facts concerning Servois are from [Bradley 2002, Boyer 1894].
In a word, I am convinced that the method of infinitesimals neither has nor can have a theory and that, in practice, it is a dangerous tool in the hands of beginners. It necessarily imprints a long-lasting character of awkwardness and pusillanimity upon the course of their research. Finally, anticipating for my own part the judgment of posterity, I dare to predict that this method will one day be accused, and with good cause, of having retarded the progress of the mathematical sciences. But I digress . . . [the italics are Servois’] [Servois 1814b]

The publication of these papers represents the high water mark of Servois’ mathematical career. In the same year, he was called back to active duty to serve in the defense of Paris in the final phase of the Napoleonic wars. Although the Parisians were outnumbered and defeated, he served with distinction and was made a knight of the Légion d’honneur. He returned to academic life at the artillerie school in Metz, but he was made curator of the artillerie museum in Paris, a military position, in June 1816. He published two further articles in the Annales in 1817 and 1826 and wrote the entry on trajectory in the Dictionaire de l’artillerie, published in 1822. He retired to his birthplace, Mont-de-Laval, in 1827, and died 20 years later.

The Essai of 1814 represents an attempt to rigorize Lagrange’s symbolical calculus, as developed by Arbogast and François, and use it as a foundation for differential calculus. As we have seen, during its prehistory, Leibniz thought of symbolic calculus as a matter of analogy, not as a legitimate method of proof. Furthermore, there is a danger of circularity in using power series as the tool for defining the derivative, as Lagrange intended. Admittedly, there was no circularity in Lagrange’s arguments, as his series expansions were derived without the use of the derivative, unlike Taylor’s. Servois uses algebraic properties of operators as the basis for a deductive system, thereby rising above the level of mere analogy and, so he hoped, avoiding circularity.

The Essai is a strikingly original piece of mathematics. As Gregory noted, it was where “the method of separation of symbols was first correctly given.” Although Servois refers to operators as “functions,” he clearly distinguishes these functions of functions from the more ordinary functions of an independent variable. Unlike Gregory, who considers an abstract class of operations satisfying certain axioms, Servois considers certain particular types of operators, notably the translation operator $E$, which we have already encountered, and the multiplication operator, which maps a function $f$ to a product $\phi f$ with specific interest in multiplication by a constant, which in turn includes the case of the identity operator. From these, other operators can be derived, notably the difference operator $\Delta = E - 1$.

Servois defines the notions of commutative and distributive operators as Gregory does, but does not mention the law of repetition. Apparently he has no need of this, as his goal is to investigate the foundations of analysis, as opposed to proving higher level theorems, as Gregory did. Like Gregory’s, Servois’ form of the commutative law allows him to get by without an associative law, which must await the work of Hamilton before being identified as an important axiom for pure mathematics. Servois demonstrates that the translation operator and the operation of multiplying by a constant are both distributive, and that any two such operators commute. By considering closure properties, he essentially shows that the set of all linear combinations of such operators forms a ring. Although Servois does not use the terms “ring” and “linear combination,” it is easy to modify his arguments in order to derive a proof that a class of distributive, commutative operators forms a ring under the operations of addition and composition. Let us pause briefly to observe that by considering distributive operators that commute with multiplication by a constant, Servois is considering what we
call linear operators. For if $F$ has these two properties, then for functions $\phi$ and $\psi$ and a constant $a$ we have

$$F(\phi + \psi) = F\phi + F\psi$$

and

$$F(a\phi) = a(F\phi).$$

After this elegant exposition, decades ahead of its time, Servois treads in decidedly more turbid waters. Following a long series of formal power series manipulations where, among other things, Newton’s finite difference formula is derived and generalized, he defines the differential as an infinitesimal (infinitinôme):

$$dz = \Delta z - \frac{1}{2} \Delta^2 z + \frac{1}{3} \Delta^3 z - \cdots$$

This is essentially to say that $dz$ is $\log(1 + \Delta z)$, an identity first derived by Lagrange in 1772, using his method of analogies. By taking this linear operator as the definition of the differential, Servois proceeds to derive the elementary differentiation formulas, which are a consequence of this definition and the algebraic properties of the operator $\Delta$.

Nowhere in his power series manipulations, nor in the consideration of $dz$ as defined above, does Servois consider such issues as convergence or the domain of functions to which such infinite series methods might apply. This, of course, represents the Achilles heel of his foundational program. By the time in the next decade that Cauchy has sorted out appropriate definitions of convergence and radius of convergence, an appropriate definition of the limit is available, also due to Cauchy, and interest in the uses of symbolical calculus for foundational purposes begins to wane, at least on the Continent.

When the Analytical Society pioneered the adoption of continental methods in calculus, they wholeheartedly endorsed an algebraic foundation for it. Lacroix’ monumental three volume *Traité du calcul différentiel et du calcul intégral*, (1797–1798, 2nd ed. 1810–1819) described an algebraic foundation, whereas his abridged single volume *Traité élémentaire du calcul différentiel et du calcul intégral* (1806) used limits. When Babbage, Peacock, and Herschel translated the latter text, Peacock annotated every reference to limits with a note deriving the same result by Lagrangian methods. For, as noted in the introduction:

*[The traité élémentaire] may be considered as an abridgement of his great work on the Differential and Integral Calculus, although in the demonstration of First Principles, he has substituted the method of limits of D’Alembert in the place of the more correct and natural method of Lagrange which was adopted in the former. [Lacroix 1816, iii, bold type added for emphasis]*

Babbage, Peacock, and Herschel wrote a number of articles using symbolic methods over the course of the following years, so that by the 1830s the stage was set for a flourish of new results involving the application of these methods to analysis. E. T. Bell describes the middle third of the 19th century in Britain as a “somewhat shady episode of symbolic methods,” which, “despite their utility . . . were scarcely reputable mathematics, because no explicit formulation of the conditions under which they give correct results accompanied
their use” (quoted in [Koppelman 1971, 188]). This seems an overly harsh assessment of Gregory’s very careful exposition in Chapter XV of Part I of the Examples, and even more so of Murphy’s remarkable “First Memoir on the Theory of Analytical Operations” [Murphy 1837]. (We note that there was no second memoir on the theory of analytical operations, although Murphy presumably had planned one.)

Robert Murphy was born to a shoemaker in Mallow, County Cork, Ireland in 1806.5 Although his father intended that he follow him in his trade, Robert’s life was changed irrevocably when, at the age of 11, he was accidentally run over by a cart. During his year-long convalescence, he discovered a remarkable talent for mathematics, initially through some mathematical problems contained in the Cork almanac. He came to the attention of a Mr. Mulcahy, who proposed mathematical problems in the newspaper to which the 12-year-old submitted creative (and anonymous) solutions. Through Mulcahy’s efforts, a local schoolmaster agreed to educate Murphy free of charge. Although Mulcahy was subsequently unsuccessful at having Murphy accepted to Trinity College, Dublin, through his efforts Murphy’s mathematical papers eventually came to the attention of Robert Woodhouse (1773–1827), fellow of Caius College, Cambridge, and Plumian professor of astronomy and experimental philosophy. We note that Woodhouse, who had been Lucasian professor of mathematics from 1820 to 1822, was the first Cambridge man to champion Lagrange’s methods, in 1803. Although he was not widely influential in the adoption of continental methods at Cambridge, he is identified [Koppelman 1971] as having prepared the way for Peacock and the Analytical Society a generation later.

Murphy came up to Caius College in October, 1825, and he was third wrangler when he took his B.A. in 1829. He became a fellow of Caius later the same year, and dean (a chapel officer) in 1831. De Morgan says he “gradually fell into dissipated habits, and in December, 1832, left Cambridge, with his fellowship under sequestration for the benefit of his creditors” [De Morgan 1846]. Although college records indicate that these “dissipated habits” included gambling, at least one writer [Smith 1984] finds evidence in Murphy’s later letters that De Morgan may have been guardedly referring to alcoholism. He lived for a while in Ireland, and then came to London in 1836. In 1834 he was elected to the Royal Society. He wrote books on electricity and magnetism (1833) and the theory of “Algebraical Equations” (1839), as well as various articles in the Cambridge Philosophical Transactions and the Philosophical Transactions of the Royal Society of London. He contracted “a disease of the lungs,” and died in March 1843.

Murphy’s “First Memoir on the Theory of Analytical Operations“ is every bit as novel for its time as Servois’ “Essai” had been 21 years earlier. Murphy clearly distinguishes between the Operations and the objects (which he calls Subjects) upon which the operations are to be performed. In addition, he uses square brackets and postfix notation for the application of an operation; there is no possibility therefore for any confusion between the category of functions and the category of operators. For example, the translation function $E$, which he denotes by $\psi$, might be applied as follows:

$$[x^n]\psi = (x + h)^n.$$  

Reading left to right, we see the subject of the operation (the function $f(x) = x^n$), the

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5 The biographical facts concerning Murphy are from [De Morgan 1846].
operation itself, and the Result of the operation. The operations with which he is primarily concerned are $\psi$, the finite difference $\Delta$, multiplication by a constant $a$, and $dx$, which denote the operation of taking the finite difference, and after dividing it by $h$, then putting $h = 0$, which is the same as finding the differential coefficient of the subject, which we may suppose to be represented by $u$, then

$$[u] dx = \frac{du}{dx}.$$

Murphy neatly sidesteps the problems of infinite series and convergence by according the differential operator a status akin to that of primitive notion. We note, however, that the description of this operator is entirely in algebraic terms, with no mention of fluxion, differential, or limit. Furthermore, we observe the use of the term “differential coefficient,” which is reminiscent of Lagrange’s conception of the derivative. Sadly, we must also observe that Murphy’s definition of $dx$ is inadequate as stated, even for the simple case of the sine function.

An advantage of postfix notation is that when multiple operations are performed, the order of their application can simply be read left to right. Murphy notes that the order of operations is frequently significant; for example, $x^2 \psi$ and $\psi x$ are different operations, since

$$[x^n] x \psi = [x^{n+1}] \psi = (x + h)^{n+1}$$
$$[x^n] \psi x = [(x + h)^n] x = x(x + h)^n$$

When the order in which two operations are applied is irrelevant, Murphy calls the composite operator free, instead of calling the constituent operators commutative with one another. Thus, $a \psi$ is free, since $[f] a \psi = [f] \psi a$ for any function $f$. If order is significant, he refers to the composite operator as fixed. The earlier example shows that $x \psi$ is fixed.

Similarly, Murphy eschews Servois’ term distributive. Foreshadowing modern usage, he calls the operator $\phi$ linear if

$$[a \pm b] \phi = [a] \phi \pm [b] \phi,$$

although the modern usage would require also that $ap$ be free, where $a$ represents multiplication by a constant $a$. (We note that a continuous operator that is linear in Murphy’s sense is also linear in the modern sense.)

Murphy derives many of the same results as Servois, with greater brevity and clarity, while avoiding all of the questionable formal manipulations involving the latter’s definition of the differential. As a work of algebra, he charts much new territory, considering the kernel of an operator (which he calls the appendage) for the first time, and examining conjugates of an operator.

10. CONCLUDING REMARKS

It would be fascinating to be a fly on the wall in Gregory’s Trinity College study, to hear what he told his students during their tutorials concerning the foundations of calculus in the late 1830s and early 1840s. This was a transitional time in both Great Britain and the Continent. De Morgan reports in the preface to his calculus text [De Morgan 1842] that
during this period no fewer than four French authors “rejected the doctrine of series, and adopted that of limits” in their treatises on calculus. He continues:

... I have therefore no occasion to argue further against the former method, which has been thus abandoned in the country which saw its birth, and will certainly lose ground in England, when it is no longer maintained by a supply from abroad of elementary treatises written on its principles. [De Morgan 1842, p. v]

Knowing this, the reader will not be surprised to learn that De Morgan employed limits in the introductory chapter of his text. However, the exposition does not use epsilons and deltas, but rather describes limits in a manner reminiscent of d’Alembert’s definition and Lacroix’ treatment in his *Traité élémentaire*.

It would be a mistake to assume that Cauchy’s treatment of the limit was immediately embraced by the mathematical community, even in his native France. Indeed, De Morgan’s preface indicates that 15 years later, the transition to a limit-based foundation for calculus was only beginning to take place in elementary texts in France. England was that much further behind in the process, still riding the crest of the new wave of algebraic formalism, inspired by the work of the Analytical Society.

What we do know is that Gregory was a staunch supporter of algebraic methods, even though the extant applications thereof are at an intermediate, as opposed to foundational, level of deduction. Furthermore, the two authors who did expound on foundational matters and who are mentioned specifically in the crucial Chap. XV of Part I of the *Examples* are proponents of algebraic methods and, in the case of Servois, a neo-Lagrangian who freely uses infinite series for the very definition of the derivative.

Gregory died tragically young. Had he lived longer, we would probably have concrete evidence of his foundational views. The evidence that does exist is consistent with a belief that he had great faith in the promise of algebraic methods. The wisdom of such faith in algebraic methods, for the benefit of mathematics generally, is indicated by the later successes, for example, of Hamilton and Cayley (the latter published three papers in the *CMJ* as an undergraduate) in the early history of abstract algebra. We believe that the young Gregory of historical record would have followed in the footsteps of Lagrange, Servois, Murphy and the Analytical Society had he actually written a treatment of the first principles of calculus, and that the hypothetical Gregory, had he lived to a ripe old age, would have gradually and sagely recognized the superior methods of Cauchy, Bolzano, and Weierstrass.

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