mathématiques en Italie (4 vols., Paris, 1838–1841), III, 158, mention the work of Tartaglia but not that of Cardan.


10. Opera omnia, IV, 558.

THE FINITE FOURIER SERIES AND ELEMENTARY GEOMETRY

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Every freshman knows, or soon learns, that the periodic sequence \( x_0 = 1 \)
\( x_1 = -1 \), \( x_2 = 1 \), \( x_3 = -1 \), \( \ldots \) has a general term \( x_n \) which may be written as \( x_n = (-1)^n \). This is a very special case of representation of a periodic sequence by the so-called finite Fourier Series. The finite Fourier series is the analogue for periodic sequences of the ordinary Fourier series expansion of periodic functions. Its only mention in our textbook literature is under the heading of practical harmonic analysis, or trigonometric interpolation, as found in books on applied mathematics or numerical methods; and yet, the range of applications of the finite Fourier series extends beyond this important practical problem. It is intimately related to the Gaussian sums and has been used for number theoretic purposes by Eisenstein and more recently by H. A. Rademacher. The following lines present the basic properties of the (complex) finite Fourier series stressing its geometric significance and followed by a few applications to extremal problems of elementary geometry.

§1. THE Finite Fourier Series

1. The finite Fourier series. We recall the general problem of polynomial interpolation: If we are given \( k \) distinct complex numbers \( \omega_0, \omega_1, \ldots, \omega_{k-1} \), and a second set of \( k \) arbitrary complex numbers \( z_0, z_1, \ldots, z_{k-1} \), then there is one and only one polynomial
\[
P(x) = \sum_{v=0}^{k-1} \xi_v x^v
\]

satisfying the equations \( P(\omega_v) = z_v \) \( (v = 0, 1, \ldots, k - 1) \) or

\[
z_v = \xi_0 + \xi_1 \omega_v + \xi_2 \omega_v^2 + \cdots + \xi_{k-1} \omega_v^{k-1} \quad (v = 0, 1, \ldots, k - 1). \quad [1]
\]

Indeed, the system of linear equations (1) in the unknowns \( \xi_v \) has a non-vanishing (Vandermonde) determinant and has therefore a unique solution. We obtain the finite Fourier series (abbreviated in the sequel to F.F.S.) if we choose the numbers \( \omega_v \) to be the \( k \)th roots of unity; so for the remainder of this paper we shall assume that

\[
\omega_v = e^{2\pi iv/k} \quad (v = 0, 1, \ldots, k - 1).
\]

This equation defines \( \omega_v \) for all integral values of \( v \), a fact occasionally used later.
The equations (1) now represent, by definition, the f.F.S. expansion of the given sequence $z_0, \cdots, z_{k-1}$, while the coefficients $\xi_0, \cdots, \xi_{k-1}$, are called the f.F. coefficients of the given sequence $(z_r)$.

In order to solve the system (1), i.e. find the f.F. coefficients $\xi_r$, a few properties of the roots of unity (2) are necessary. Thus (2) readily implies such relations as

$$
\omega^\alpha_r = \omega^\alpha_{\alpha r}, \quad \omega^\omega_r = \omega^-\omega_r, \quad \omega^\omega_1 = \omega_\omega, \quad \omega_{\alpha+1} = \omega_\omega \omega_1,
$$

which will be used freely. Especially important for us are the so-called orthogonality relations

$$
\sum_{r=0}^{k-1} \omega^{-\alpha}_r \omega^{-\beta}_r = \begin{cases} k & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases} \quad (\alpha, \beta = 0, 1, \cdots, k - 1).
$$

They allow us to determine the f.F. coefficients as follows: If we multiply (1) or

$$
z_r = \sum_{\mu=0}^{k-1} \xi_{\mu} \omega^\mu_r
$$

by $\omega^\omega_\omega = \omega^\omega_\omega$ and sum over all $\nu$, we find

$$
\sum_{r=0}^{k-1} z_r \omega^\omega_r = \sum_{\mu, \nu} \xi_{\mu} \omega^\mu_r \omega^{-\nu}_r = \sum_{\omega=0}^{k-1} \omega_\omega \sum_{\mu=0}^{k-1} \omega^\mu \omega^{-\nu} = \xi_\omega \omega_\omega \omega_1
$$

by (4). The f.F. coefficients are therefore given by the relations

$$
\xi_\omega = \frac{1}{k} \left( z_0 + z_1 \omega_\omega + z_2 \omega^2_\omega + \cdots + z_{k-1} \omega^{k-1}_\omega \right), \quad (\alpha = 0, 1, \cdots, k - 1).
$$

2. Geometric interpretation. It is both suggestive and desirable for future purposes to plot the numbers $z_r$ in the complex plane where they may be thought of as the successive vertices of a closed polygon $\Pi = (z_r)$. In particular, the sequence $(\omega_\omega)$ represents there a regular $k$-gon $\Pi_1 = (\omega_\omega)$, while the sequence $(\omega^\omega_\omega)$ represents a $k$-gon $\Pi_\omega = (\omega^\omega_\omega)$ obtained from $\Pi_1$ by starting from $\omega_\omega = 1$ and taking every $\alpha$th vertex. $\Pi_\alpha$ is a starred regular $k$-gon if and only if $\alpha$ is relatively prime to $k$; otherwise it is a regular polygon of lower order described several times so as to result in a total of $k$ vertices. We also include the polygon $\Pi_0 = (\omega^\omega_\omega)$ all of whose $k$ vertices coincide with $\omega_\omega = 1$. A glance at the f.F.S. (1) now shows that our polygon $\Pi = (z_r)$ has been analyzed harmonically, i.e. has been represented as a linear combination of the fundamental $k$-gons $\Pi_\alpha$ in a manner suggested by the symbolic relation $\Pi = \xi_0 \Pi_0 + \xi_1 \Pi_1 + \cdots + \xi_{k-1} \Pi_{k-1}$.

It is sometimes desirable to re-write the expansion (1) by grouping its terms as follows:

$$
z_r = \xi_0 + (\xi_1 \omega_r + \xi_{k-1} \omega^{k-1}_r) + (\xi_2 \omega^2_r + \xi_{k-2} \omega^{k-2}_r) + \cdots,
$$
or
\[
\zeta_\nu = \xi_0 + (\xi_1 \omega_\nu + \xi_{k-1} \bar{\omega}_\nu) + (\xi_2 \omega_\nu^2 + \xi_{k-2} \bar{\omega}_\nu^2) + \ldots, \tag{6}
\]
where the last term is the central term of (1), or the sum of the two central terms of (1), depending on whether \(k\) is even or odd. Also the individual terms of (6) have a simple geometric interpretation which I learned orally from Fritz John. Let us look at the first term
\[
\zeta_\nu = \xi_1 \omega_\nu + \xi_{k-1} \bar{\omega}_\nu, \quad (\nu = 0, \ldots, k - 1). \tag{7}
\]
Separating reals and imaginaries by setting
\[
z_\nu = x_\nu + i y_\nu, \quad \xi_1 = a_1 + ib_1, \quad \xi_{k-1} = c_1 + id_1, \quad \omega_\nu = \cos \frac{2\pi \nu}{k} + i \sin \frac{2\pi \nu}{k},
\]
we find the coordinates of the point (7) to be
\[
x_\nu = (a_1 + c_1) \cos \frac{2\pi \nu}{k} + (-b_1 + d_1) \sin \frac{2\pi \nu}{k},
\]
\[
y_\nu = (b_1 + d_1) \cos \frac{2\pi \nu}{k} + (a_1 - c_1) \sin \frac{2\pi \nu}{k}.
\]
If we write
\[
a' = a_1 + c_1, \quad b' = -b_1 + d_1, \quad c' = b_1 + d_1, \quad d' = a_1 - c_1,
\]
it is clear that the polygon \(\Pi'_1\), of vertices (7), is the image of the regular \(k\)-gon \(\Pi_1 = (\omega_\nu)\) by the affine transformation
\[
x' = a' x + b' y, \tag{8}
\]
\[
y' = c' x + d' y.
\]
Conversely, by following our relations backwards we see that we may start with an arbitrary affine transformation (8) and find that the image of \(\Pi_1\), by (8), may be written in the form (7). Generally, if \(\alpha\) is any integer in the range \(1 \leq \alpha \leq k/2\), we find the \(\alpha\)th term of (6), or
\[
z^{(\alpha)}_\nu = \xi_1 \omega^{(\alpha)}_\nu + \xi_{k-1} \bar{\omega}^{(\alpha)}_\nu, \quad (\nu = 0, \ldots, k - 1), \tag{9}
\]
to represent the image \(\Pi^{(\alpha)}\), of \(\Pi_\alpha = (\omega^{(\alpha)}_\nu)\), by a uniquely defined affine transformation
\[
x^{(\alpha)} = a^{(\alpha)} x + b^{(\alpha)} y, \tag{10}
\]
\[
y^{(\alpha)} = c^{(\alpha)} x + d^{(\alpha)} y,
\]
whose coefficients are related to those of (9) by precisely the same relations which connected (8) to (7).
We may state our result as follows: The expansion (6) represents our given polygon $\Pi$ as a uniquely defined sum of a constant $\xi_0$ (the centroid of $\Pi$) plus $[k/2]$ $k$-gons $\Pi^{(a)}_\alpha (\alpha = 1, \ldots, [k/2])$, where $\Pi^{(a)}_\alpha$ is an affine image of the fundamental $k$-gon $\Pi_\alpha = (\omega_\alpha^\alpha)$. [2]

The expansion (6) should be called the real f.F.S. of $(z_r)$, in contradistinction to (1) which is the complex f.F.S. These terms are justified by the following remarks:

1. If all $z_r$ are real then $\xi_0$ is real and $\xi_{k-a} = \overline{\xi_a}$ ($\alpha = 1, 2, \ldots, k-1$), and conversely. Indeed, the direct statement follows from (5), the converse one from (6).

2. If the $z_r$ are real then all terms of the real f.F.S. (6) are real. Evidently so, because $\xi_\alpha \omega_\beta^\alpha + \xi_{k-a} \omega_\alpha^\alpha$ is the sum of two complex conjugate terms. The same result can be seen from (9) which turns out to be a degenerate affine transformation with $y^{(a)} = 0$, i.e. mapping the whole plane onto the real axis.

3. The analogue of the Parseval relation. [3] Besides the f.F.S.

\[ z_r = \sum_{\alpha=0}^{k-1} \xi_\alpha \omega_\alpha^\alpha, \quad (r = 0, \ldots, k-1), \]

let

\[ x_r = \sum_{\beta=0}^{k-1} \xi_\beta \omega_\beta^\beta, \quad (r = 0, \ldots, k-1), \]

be the f.F.S. of a second sequence $(x_r)$. We now find

\[ \sum_{r=0}^{k-1} z_r x_r = \sum_{\alpha, \beta} \xi_\alpha \omega_\alpha^\alpha \xi_\beta \omega_\beta^\beta = \sum_{\alpha, \beta} \xi_\alpha \omega_\alpha^\alpha \sum_{\beta} \omega_\beta \omega_\beta^\beta, \]

and therefore, by the orthogonality relations (4),

\[ \sum_{r=0}^{k-1} z_r x_r = k \cdot \sum_{\alpha=0}^{k-1} \xi^\alpha \omega^\alpha. \]

In particular, if $x_r = z_r$ we find that

\[ \sum_{r=0}^{k-1} |z_r|^2 = k \cdot \sum_{\alpha=0}^{k-1} |\xi^\alpha|^2. \]

This is the finite Parseval relation which we shall use repeatedly.

4. The convolution of sequences. Let again $(z_r)$ and $(x_r)$ be the two sequences (11) and (12). Out of them we form a new sequence $(z'_r)$ defined by the equations

\[ z'_r = \sum_{\alpha=0}^{k-1} z_{\alpha} x_{r-\alpha}, \quad (r = 0, \ldots, k-1), \]
where we observe the convention of treating \((x_r)\) as an infinite sequence of period \(k\), i.e. \(x_{-1} = x_{k-1},\ x_{-2} = x_{k-2}\), and so forth. In explicit form, the relations (14) are

\[
\begin{align*}
z_0' &= z_0 x_0 + z_1 x_{k-1} + z_2 x_{k-2} + \cdots + z_{k-1} x_1 \\
z_1' &= z_0 x_1 + z_1 x_0 + z_2 x_{k-1} + \cdots + z_{k-1} x_2 \\
&\vdots \\
z_{k-1}' &= z_0 x_{k-1} + z_1 x_{k-2} + z_2 x_{k-3} + \cdots + z_{k-1} x_0.
\end{align*}
\]

(15)

We say that \((s_r')\) is obtained from the sequences \((s_r)\) and \((x_r)\) by convolution, an operation which we indicate by the symbolic relation \((s_r') = (s)_r (x_r)\). The f.F.S. of \((s_r')\) is readily found as follows: From (14), (11) and (12) we have

\[
z_r' = \sum_a z_a x_{r-a} = \sum_{\lambda, \mu, \alpha} \zeta_\lambda \omega^\mu \xi_\alpha \omega^{-\alpha} = \sum_{\lambda, \mu, \alpha} \zeta_\lambda \xi_\mu \omega^\mu \lambda^{-\mu} \omega^{-\alpha}
\]

\[
= \sum_{\lambda, \mu} \zeta_\lambda \xi_\mu \omega^\mu \sum_{a=0}^{k-1} \omega^{a\mu} \omega^{-\alpha},
\]

and finally, by (4),

\[
z_r' = \sum_{\lambda=0}^{k-1} (\zeta_\lambda \xi, k) \omega^\lambda \quad (\nu = 0, \cdots, k - 1).
\]

This being the f.F.S. of \((s_r')\), we see that the f.F. coefficients \((\zeta_r')\) of the convolution \((s_r') = (s)_r (x_r)\) are given by

\[
\zeta_r' = \zeta_r \xi, k \quad (\nu = 0, \cdots, k - 1).
\]

We wish to state this result in a somewhat more convenient form. We may look on the convolution operation \((s_r') = (s)_r (x_r)\) as the linear transformation (15) from the variables \((s_r)\) to the new variables \((s_r')\). (15) may be described as a cyclic transformation because the successive rows of its matrix are obtained from its first row by successive cyclic permutations. Now (16) shows that the f.F. coefficients of \((s_r')\) are obtained by multiplying those of \((s_r)\), i.e. the \((\zeta_r)\), by the factors

\[
\xi_r, k = x_0 + x_1 \omega_r + x_2 \omega_r^2 + \cdots + x_{k-1} \omega_r^{k-1}
\]

which we prefer to write in the equivalent form

\[
\xi_r, k = x_0 + x_{k-1} \omega_r + x_{k-2} \omega_r^2 + \cdots + x_1 \omega_r^{k-1}.
\]

If we change our notation by defining \(a_0 = x_0, a_1 = x_{k-1}, a_2 = x_{k-2}, \cdots, a_{k-1} = x_1\), we may state our result as
Theorem 1. If \((\xi_v)\) are the f.F. coefficients of a sequence \((z_v)\), and if we subject the sequence \((z_v)\) to the cyclic transformation

\[
\begin{align*}
z_0' &= a_0z_0 + a_1z_1 + a_2z_2 + \cdots + a_{k-1}z_{k-1} \\
z_1' &= a_{k-1}z_0 + a_0z_1 + a_1z_2 + \cdots + a_{k-2}z_{k-1} \\
&\vdots \\
z_{k-1}' &= a_{k-2}z_0 + a_{k-3}z_1 + a_0z_2 + \cdots + a_{k-3}z_{k-2}
\end{align*}
\]

(17)

then the f.F. coefficients of the new sequence \((z_v')\) are

\[
\xi_v' = \xi_v f(\omega_v) \quad (v = 0, \cdots, k - 1),
\]

(18)

where

\[
f(z) = a_0 + a_1z + \cdots + a_{k-1}z^{k-1},
\]

(19)

and will be referred to as the representative polynomial of the cyclic transformation (17). [4]

By the finite Parseval relation (13) we have

\[
\sum |z_v'|^2 = k \sum |\xi_v'|^2.
\]

If we substitute on the right-hand side the values (18) we obtain the following

Corollary. The relations (17) imply the identity

\[
\sum_{v=0}^{k-1} |z_v'|^2 = k \sum_{v=0}^{k-1} |f(\omega_v)|^2 |\xi_v|^2,
\]

(20)

where \((\xi_v)\) are the f.F. coefficients of the sequence \((z_v)\), while \(f(z)\) is defined by (19).

The relation (20) should be regarded as an identity in the \(k\) variables \((z_v)\), the \((\xi_v)\) being given in terms of the \((z_v)\) by (5). It has a number of applications to elementary geometry to which we now turn.

§2. Geometric Applications

5. The MONTHLY Problem no. 3547. The problem referred to was proposed by Martin Rosenman in 1932 and may be stated as follows:

Let \(\Pi = P_0P_1 \cdots P_{k-1}\) be a closed polygon in the plane \((k \geq 2)\). Denote by \(P_0', P_1', \cdots, P_{k-1}'\), the midpoints of the sides \(P_0P_1, P_1P_2, \cdots, P_{k-1}P_0\), respectively, obtaining the first derived polygon \(\Pi' = P_0'P_1' \cdots P_{k-1}'\). Repeat the same construction on \(\Pi'\) obtaining the second derived polygon \(\Pi''\), and finally, after \(n\) constructions, obtain the \(n\)th derived polygon \(\Pi^{(n)} = P_0^{(n)}P_1^{(n)} \cdots P_{k-1}^{(n)}\). Show that the vertices of \(\Pi^{(n)}\) converge, as \(n \to \infty\), to the centroid of the original points \(P_0, P_1, \cdots, P_{k-1}\). [5].
Let \( z_0, \ldots, z_{k-1} \) denote the complex coordinates of the vertices of \( \Pi \). It is then clear that the coordinates of the vertices of the polygon \( \Pi' \) are obtained from \((z_r)\) by means of a linear cyclic transformation (17) of coefficients \( a_0 = a_1 = 1/2, a_2 = \cdots = a_{k-1} = 0 \). Its representative polynomial (19) now becomes
\[
f(z) = (1 + z)/2,
\]
and by (20) we have the identity
\[
\sum_{r=0}^{k-1} |z_r'|^2 = k \cdot \sum_{r=0}^{k-1} \left| \frac{1 + \omega_r}{2} \right|^2 \cdot |\xi_r|^2,
\]
where \((\xi_r)\) are the f.F. coefficients of \((z_r)\). We now make the further assumption
\[
\sum_{r=0}^{k-1} z_r = 0,
\]
which means that the centroid of the vertices of \( \Pi \) is at the origin. By (5), our assumption (22) is equivalent to \( \xi_0 = 0 \), so that (21) now implies
\[
\sum_{r=0}^{k-1} |z_r'|^2 = k \cdot \sum_{r=1}^{k-1} \left| \frac{1 + \omega_r}{2} \right|^2 \cdot |\xi_r|^2 \leq \max_{r=1, \cdots, k-1} \left| \frac{1 + \omega_r}{2} \right|^2 \cdot k \sum_{r=1}^{k-1} |\xi_r|^2
\]
\[
= \cos^2 \frac{\pi}{k} \sum_{r=0}^{k-1} |z_r|^2.
\]
In justifying the last equality sign of (23) we have used firstly, the finite Parseval relation (13) with \( \xi_0 = 0 \), secondly, that the maximum occurring in (23) is \( \cos^2 (\pi/k) \). This last fact is immediately seen geometrically, because the points \((1 + \omega_r)/2\), which are the midpoints of the segments joining \( z = 1 \) to \( z = \omega_r \), are the vertices of a regular polygon inscribed in the circle having the segment joining \( z = 0 \) to \( z = 1 \) as diameter. This proves the following

**Theorem 2.** Let \( z_0, z_1, \ldots, z_{k-1} \) be the vertices of a \( k \)-gon such that \( \sum z_r = 0 \), and let \( z_0', z_1', \ldots, z_{k-1}' \) be the midpoints of its sides, then
\[
\sum_{r=0}^{k-1} |z_r'|^2 \leq \cos^2 \frac{\pi}{k} \cdot \sum_{r=0}^{k-1} |z_r|^2,
\]
with equality if and only if
\[
z_r = \xi_0 + \xi_{k-1} + \xi_r = \xi_1 \omega_r + \xi_{k-1} \omega_r, \quad (r = 0, \ldots, k - 1),
\]
i.e. when our polygon \( \Pi \) is an affine transform of an ordinary regular \( k \)-gon with its center at the origin.

The inequality (24) follows from (23). To find the cases when we have the equality sign in (24) we need the further obviously geometric fact that the equality
\[
\left(\frac{1 + \omega_r}{2}\right)^2 = \max_{r=1, \ldots, k-1} \left(\frac{1 + \omega_r}{2}\right)^2 = \cos^2 \frac{\pi}{k}
\]

is true only if \( \nu = 1 \), or \( \nu = k - 1 \). Now we have equality in (24), i.e. in (23), if and only if \( \zeta_r = 0 \) for \( \nu = 2, 3, \ldots, k-2 \), which means that the f.F.S. of \( \{z_r\} \) indeed reduces to (25).

As special cases we mention the following: 1. If \( k = 3 \) we always have equality in (24). 2. If \( k = 4 \) we have equality in (24) if and only if the quadrilateral \( \Pi \) is a parallelogram.

A solution of Problem no. 3547 is now immediate; indeed, it is clear from (18) for \( \nu = 0 \) (or also directly) that we have \( \zeta_0 = 0 \), hence \( \sum z_r = 0 \). We may therefore apply the inequality (24) to any pair of consecutive derived polygons \( \Pi^{(e-1)} \), \( \Pi^{(e)} \), obtaining

\[
\sum_{r=0}^{k-1} |z_r^{(e)}|^2 = \cos \frac{2\pi}{k} \cdot \sum_{r=0}^{k-1} |z_r^{(e-1)}|^2.
\]

From these equations for \( s = 1, 2, \ldots, n \) we obtain

\[
\sum_{r=0}^{k-1} |z_r^{(n)}|^2 \leq \cos \frac{2n\pi}{k} \cdot \sum_{r=0}^{k-1} |z_r|^2.
\]

As \( n \to \infty \), the right-hand side tends to zero, because \( \cos (\pi/k) < 1 \), showing that

\[
\lim_{n \to \infty} z_r^{(n)} = 0, \quad (\nu = 0, \ldots, k - 1),
\]

as required by the problem.

Our last inequality (26) gives some idea of the rapidity of convergence and of its deterioration as \( k \) increases. We actually have equality in (26) for all \( n \) if and only if \( \Pi \) is an affine image of a regular polygon. More precise information concerning the sequence of polygons \( \Pi^{(n)} \) is furnished by (18). For instance if \( k = 4 \), then \( f(\omega_2) = (1+\omega_2)/2 = 0 \), therefore \( \Pi' \) and all succeeding \( \Pi^{(n)} \) are parallelograms, a fact which is evident geometrically.

6. A generalization of Problem no. 3547. Our generalization is as follows. We start as before from a polygon \( \Pi = \{z_r\} \) and derive from it the new polygon \( \Pi' = \{z'_r\} \), not by the special midpoint construction as heretofore, but by means of a fixed cyclic transformation (17). Let again \( \Pi^{(n)} \) be the \( n \)th polygon derived by repeating the construction (17) \( n \) times. Under what circumstances will the vertices \( P_0^{(n)}, \ldots, P_{k-1}^{(n)} \) of the polygon \( \Pi^{(n)} \) converge to limit positions for an arbitrary choice of the original polygon \( \Pi? \) [6]

This question is readily settled if we employ the finite Fourier series of our successive polygons and Theorem 1. Indeed, if \( \{\zeta^{(n)}\} \) are the f.F. coefficients of \( \Pi^{(n)} \) \((n = 0, 1, \ldots)\), we find by a repeated application of (18) that

\[
\zeta_r^{(n)} = (f(\omega_r))^n \zeta_r, \quad (\nu = 0, \ldots, k - 1).
\]
Therefore the \( \lim \xi^{(n)} \), as \( n \to \infty \), will exist for all \( \nu \) and arbitrary values of \( (\xi) \) if and only if, for each \( \nu \), we have either \( f(\omega_\nu) = 1 \), or else \( |f(\omega_\nu)| < 1 \). We may state our result as

**Theorem 3.** Let \( \Pi = (z_\nu) \) be a plane \( k \)-gon and let \( \Pi' = (z'_\nu) \) be the polygon derived from \( \Pi \) by the construction described by the cyclic linear transformation (17). Let \( \Pi'' = (z''_\nu) \) be the polygon obtained from \( \Pi' \) by the same construction (17) and finally, let \( \Pi^{(n)} = (z^{(n)}_\nu) \) be the polygon obtained after \( n \) iterations of the construction (17). The \( k \) limits

\[
\lim_{n \to \infty} z^{(n)}_\nu = z^{(n)}_\nu, \quad (\nu = 0, \ldots, k - 1),
\]

will exist for every polygon \( \Pi \) if and only if all of the \( k \) roots of unity \( (\omega_\nu) \) fall into two classes \( (\omega_\rho) \) and \( (\omega_\nu) \) such that

\[
f(\omega_\rho) = 1 \quad \text{for the roots of the first class \((\omega_\rho)\),}
\]

\[
|f(\omega_\nu)| < 1 \quad \text{for the roots of the second class \((\omega_\nu)\).}
\]

If these conditions are satisfied, then the vertices of the limit polygon are given by

\[
z^{(n)}_\nu = \sum_{(\nu)} \xi_\nu \omega_\nu^\rho \quad (\nu = 0, \ldots, k - 1),
\]

which is the sum of precisely those terms of the f.F.S. (1) of the original polygon \( \Pi = (z_\nu) \) which correspond to the first class of subscripts \( (\rho) \) according to (29). In particular, the limit of \( \Pi^{(n)} \) is the centroid of the vertices of \( \Pi \) if and only if the f.F.S. (30) reduces to the constant term \( \xi_0 \) of (1), which is the case if and only if

\[
f(1) = 1, \quad |f(\omega_\nu)| < 1 \quad \text{if} \quad \nu = 1, 2, \ldots, k - 1.
\]

This theorem has already been established up to the conditions (29) inclusive. If they are verified we get, letting \( n \to \infty \) in (27), the values

\[
\xi^{(n)}_\rho = \xi_\rho, \quad \xi^{(n)}_\nu = 0,
\]

for the f.F. coefficients of the limit polygon \( \Pi^{(\infty)} \). But then the f.F.S. of \( \Pi^{(\infty)} \) is indeed (30).

**7. An extremal problem.** The argument which led from the identity (20) to Theorem 2 may be readily extended so as to furnish a general inequality, of the same type as (24), in terms of a general cyclic transformation (17). We prefer, however, to discuss only one further special case. Starting as before from the polygon \( \Pi = (z_\nu) \), we consider the special cyclic transformation

\[
z'_\nu = z_{\nu+1} - z_\nu \quad (\nu = 0, \ldots, k - 1; z_k = z_0).
\]

This is indeed identical with (17) if \( a_0 = -1, a_1 = 1, a_2 = \ldots = a_{k-1} = 0 \), hence

\[
f(z) = -1 + z.
\]
But then (20) reduces to the identity

\[ \sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 = k \cdot \sum_{\nu} |\omega_{\nu} - 1|^2 \cdot |\xi_{\nu}|^2. \]  

(33)

From this we draw two conclusions: Firstly

\[ \sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 = k \sum_{\nu} |\omega_{\nu} - 1|^2 \cdot |\xi_{\nu}|^2 \]

\[ \leq \left( \max_{\nu} |\omega_{\nu} - 1|^2 \right) \cdot k \sum_{\nu} |\xi_{\nu}|^2 \]

(34)

\[ = \left( \max_{\nu} |\omega_{\nu} - 1|^2 \right) \cdot \sum_{\nu} |z_{\nu}|^2, \]

by the finite Parseval relation. Secondly, assuming that \( \sum z_{\nu} = 0 \), or \( \xi_0 = 0 \), we have

\[ \sum_{\nu} |z_{\nu+1} - z_{\nu}|^2 = k \cdot \sum_{\nu=1}^{k-1} |\omega_{\nu} - 1|^2 \cdot |\xi_{\nu}|^2 \]

(35)

\[ \geq \left( \min_{\nu=1, \ldots, k-1} |\omega_{\nu} - 1|^2 \right) \cdot \sum_{\nu=0}^{k-1} |z_{\nu}|^2. \]

This proves

\textbf{Theorem 4.} Let \( \Pi = P_0P_1 \cdots P_{k-1} \) be a plane closed \( k \)-gon having the point 0 as centroid of its vertices. The following inequalities then hold

\[ 4 \sin^2 \frac{\pi}{k} \leq \frac{\sum_{\nu} (P_\nu P_{\nu+1})^2}{\sum_{\nu} (OP_\nu)^2} \leq 4 \sin^2 \left( \frac{\pi}{k} \left[ \frac{k}{2} \right] \right), \]

(36)

where we have equality on the left side if and only if \( \Pi \) is an affine image of an ordinary regular \( k \)-gon; the equality sign holds on the right side under circumstances which depend on the parity of \( k \) as follows:

1. If \( k = 2p \), we have equality on the right if and only if

\[ P_0 = P_2 = \cdots = P_{2p-2} \text{ and } P_1 = P_3 = \cdots = P_{2p-1}, \]

i.e. our polygon is a segment described to and fro \( p \) times. The right-hand side of (36) reduces to the value 4.

2. If \( k = 2p + 1 \), we have equality on the right if and only if \( \Pi \) is an affine image of that starred regular \( k \)-gon, inscribed in the unit circle, which has the largest side among all such starred regular \( k \)-gons. The right-hand side of (36) reduces to the value \( 4 \cos^2 \left( \frac{\pi}{2k} \right) \).
The conditions of equality in (36) follow by inspection of the multiple relations (34) and (35). Thus the extreme right member of (36) is the value of the maximum occurring in (34). Equality occurs on the right in (36) if and only if we have equality in (34). If \( k = 2p \), this is the case if and only if \( z_0 = \cdots = z_{p-1} = z_{p+1} = \cdots = z_{k-1} = 0 \), i.e. the f.F.S. of \( \Pi \) reduces to

\[ z_r = z_p \omega_r^p = z_p ( -1 )^r. \]

If \( k = 2p + 1 \) we have equality in (34) if and only if the f.F.S. of \( \Pi \) reduces to

\[ z_r = z_p \omega_r^p + z_{p+1} \omega_r^{p+1} = z_p \omega_r^p + z_{p+1} \omega_r^p, \]

which is an affine image of the starred polygon \( \Pi_p = ( \omega_r^p ) \) described in the theorem (compare with (9)). The conditions of equality on the left side of (36) are obtained by a similar argument involving (35).

A few special cases are worth mentioning separately:

If \( k = 3 \), the extreme members of (36) are equal and are 3, whence the identity

\[ \sum_{r=0}^{2} (P_r P_{r+1})^2 = 3 \cdot \sum_{r=0}^{2} (OP_r)^2. \]

If \( k = 4 \) we have

\[ 2 \leq \frac{\sum_{r=0}^{3} (P_r P_{r+1})^2}{\sum_{r=0}^{3} (OP_r)^2} \leq 4, \]

with equality on the left if and only if \( P_0 P_1 P_2 P_3 \) is a parallelogram, and equality on the right if and only if \( P_0 = P_2, P_1 = P_3 \).

8. The area of polygons. Let \( A \) denote the oriented area of the polygon \( \Pi = (z_0, z_1, \cdots, z_{k-1}) \). Let \( z_r = x_r + iy_r \); then

\[ \text{Area of triangle } O z_r z_{r+1} = \frac{1}{2} (x_r y_{r+1} - x_{r+1} y_r) = \frac{1}{2} \Im(\bar{z}_r z_{r+1}). \]

The sum of the areas of these \( k \) triangles being the area of the \( k \)-gon, we have

\[ A = \frac{1}{2} \sum_{r=0}^{k-1} \Im(\bar{z}_r z_{r+1}) = \frac{1}{4i} \sum_{r=0}^{k-1} (\bar{z}_r z_{r+1} - z_r \bar{z}_{r+1}). \]

The area \( A \) may now readily be expressed in terms of the f.F. coefficients of \( \Pi \). Indeed, by (1) we have

\[ \sum_r \bar{z}_r z_{r+1} = \sum_{a, \beta} \xi_a \omega_r^a \eta_r \omega_{r+1}^\beta = \sum_{a, \beta} \xi_a \xi_r^a \omega_r^\beta \sum_r \omega_r \omega_r^p \]

and by the orthogonality relations (4) we get
\[ \sum_r \xi_r \omega_{r+1} = k \cdot \sum_a |\xi_a|^2 \omega_a. \]

Taking the imaginary parts we obtain, in view of (37), the final expression

\[ A = \frac{k}{2} \sum_{a=0}^{k-1} |\xi_a|^2 \sin \frac{2\pi a}{k}. \]

This remarkable formula reveals a few facts at a glance:

1. The area \( A \) is equal to the sum of the oriented areas of the \( k-1 \) regular \( k \)-gons \( \xi_1 \Pi_1, \xi_2 \Pi_2, \ldots, \xi_{k-1} \Pi_{k-1} \) into which \( \Pi \) is analyzed by its finite Fourier series (1). For indeed, the oriented area of the polygon \( \xi_a \Pi_a = (\xi_a \omega_a) \) is visibly equal to

\[ k \cdot \frac{1}{2} |\xi_a| \cdot |\xi_a| \sin \frac{2\pi a}{k} = \frac{k}{2} |\xi_a|^2 \sin \frac{2\pi a}{k}, \]

which is the general term of (38). A second proof is obtained if we observe that \( A \) must reduce to the area of \( \xi_0 \Pi_0 \) if we assume in (38) that all \( \xi_a = 0 \) except \( \xi_0 \).

2. The area \( A \) depends only on the absolute values \( |\xi_a| \) of the F.F. coefficients of \( \Pi \) and not on their arguments.

We may finally derive an extremal property of the area. Indeed, from (38) we obtain by a now familiar argument

\[ A = \frac{k}{2} \sum_a |\xi_a|^2 \sin \frac{2\pi a}{k} \leq \left( \max_a \frac{1}{2} \sin \frac{2\pi a}{k} \right) \cdot k \sum_a |\xi_a|^2 \]

\[ = \left( \max_a \frac{1}{2} \sin \frac{2\pi a}{k} \right) \cdot \sum_0^{k-1} |z_r|^2. \]

If we set

\[ \gamma = \frac{1}{2} \max_a \sin \frac{2\pi a}{k}, \]

we may conclude that

\[ A \leq \gamma \sum_0^{k-1} |z_r|^2, \]

where we have equality if and only if we have equality in (39), which is the case if \( \xi_a = 0 \), except for those values of \( a \) for which \( \frac{1}{2} \sin \omega_a = \gamma \). This proves

**Theorem 5.** Let \( \Pi = P_0 P_1 \cdots P_{k-1} \) be a closed \( k \)-gon in the oriented plane with not all vertices coalescent, \( O \) a further point in the plane. \( A \) being the oriented area of \( \Pi \), consider the ratio

\[ R = \frac{A}{\sum (OP_r)^2}. \]
There are three different cases

1. If \( k = 4p \) then

\[
R \leq \frac{1}{2},
\]

with equality if and only if our polygon is \( z_r = \xi_r \omega^p = \xi_r \omega^{p+1} \), with \( O \) at the origin, i.e. \( \Pi \) is a square of center \( O \) described \( p \) times.

2. If \( k = 4p \pm 1 \), we have

\[
R \leq \frac{1}{2} \sin \frac{2\pi p}{k} = \frac{1}{2} \cos \frac{\pi}{2k},
\]

with equality if and only if \( \Pi \) is the starred \( k \)-gon \( z_r = \xi_r \omega^p \), and \( O \) is its center.

3. If \( k = 4p + 2 \), we have

\[
R \leq \frac{1}{2} \sin \frac{2\pi p}{k} = \frac{1}{2} \cos \frac{\pi}{k},
\]

with equality if and only if \( \Pi \) is of the form \( z_r = \xi_r \omega^p + \xi_{r+1} \omega^{p+1} \), with \( O \) at its center.

In the particular cases when \( k = 3 \) and \( k = 4 \), we have the following statements:

Let \( P_0P_1P_2 \) be a triangle, \( O \) a point in its plane, then

\[
\text{Area of } P_0P_1P_2 \leq \frac{\sqrt{3}}{4} ((OP_0)^2 + (OP_1)^2 + (OP_2)^2),
\]

with equality if and only if the triangle is equilateral with \( O \) at its center.

Let \( P_0P_1P_2P_3 \) be a quadrilateral, \( O \) a point in its plane, then

\[
\text{Area of } P_0P_1P_2P_3 \leq \frac{1}{2} ((OP_0)^2 + (OP_1)^2 + (OP_2)^2 + (OP_3)^2),
\]

with equality if and only if the quadrilateral is a square with \( O \) at its center.

9. The isoperimetric inequality for equilateral polygons. We shall now assume our polygon \( \Pi = (z_0, z_1, \ldots, z_{k-1}) \) to be equilateral, which means that

\[
|z_{r+1} - z_r| = a, \quad (\nu = 0, \ldots, k - 1).
\]

If we denote by \( L = ka \) its perimeter, then

\[
\sum_{\nu} |z_{r+1} - z_r|^2 = ka^2 = L^2/k,
\]

so that by (33) we obtain

\[
L^2 = \sum_{r=1}^{k-1} 4k^2 \sin^2 \frac{\pi \nu}{k} |z_r|^2.
\]

On the other hand, by multiplying (38) by \( 4k \tan(\pi/k) \) we have
\[
\left(4k \tan \frac{\pi}{k}\right) A = \sum_{r=1}^{k-1} 4k^2 \tan \frac{\pi}{k} \sin \frac{\pi r}{k} \cos \frac{\pi r}{k} \left| \xi_r \right|^2.
\]

Notice that the coefficients of \( \left| \xi_1 \right|^2 \) in these two expansions agree; hence if we subtract these two equations, those terms cancel and we obtain
\[
L^2 - \left(4k \tan \frac{\pi}{k}\right) A = \sum_{r=2}^{k-1} 4k^2 \sin \frac{\pi r}{k} \left( \sin \frac{\pi r}{k} - \tan \frac{\pi r}{k} \cos \frac{\pi r}{k} \right) \left| \xi_r \right|^2.
\]

Since all coefficients on the right-hand side are positive, this relation proves the following last

**Theorem 6 (Due to J. Steiner).** If \( \Pi = P_0 P_1 \cdots P_{k-1} \) is a plane equilateral closed \( k \)-gon of area \( A \) and perimeter \( L \) then

\[(40) \quad L^2 - \left(4k \tan \frac{\pi}{k}\right) A \geq 0,
\]

with equality if and only if \( \Pi \) is an ordinary regular \( k \)-gon.

The inequality (40) is called isoperimetric because it implies that among all equilateral \( k \)-gons of given perimeter \( L \) (and therefore having all sides equal to \( L/k \)), the regular \( k \)-gon has the largest area. [7]

Notes and References

1. The significance of the finite Fourier series (1) for the geometry of the triangle \( (k=3) \) is briefly discussed in the instructive book of Ernesto Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung, Leipzig, 1904.

2. We denote, as usual, by \([x]\) the greatest integer not exceeding \( x \).

3. For purposes of motivation we use here and in the next paragraph terms which are familiar from the theory of the Fourier series, a fact which of course in no way presupposes a knowledge of Fourier series theory.

4. This result may also be phrased in terms of matrices and linear transformations as follows: If we denote the linear transformations (1) and (17) by \((x) = \Omega (\xi)\) and \((x') = A (x)\) respectively, then the matrix \( \Omega^{-1} A \Omega \) assumes the diagonal form

\[
\Omega^{-1} A \Omega = \begin{pmatrix}
0 & \cdots & 0 \\
0 & f(\omega_l) & \cdots \\
\vdots & \vdots & \ddots \\
0 & 0 & \cdots & f(\omega_{k-1})
\end{pmatrix},
\]

a fact which is readily verified directly. In this form our result is substantially equivalent to a theorem of Spottiswoode on cyclic determinants; see Th. Muir, Theory of Determinants, vol. 2, p. 405. I owe this last reference to Alfred Brauer.


6. Stated in terms of matrices, our question is to decide if the \( n \)th power \( A^n \), of a given cyclic matrix \( A \), tends to a limit as \( n \to \infty \). This problem was recently solved for an arbitrary square matrix \( A \) of complex elements by R. Oldenburger, Infinite powers of matrices and characteristic roots, Duke Mathematical Journal, vol. 6, 1940, pp. 357–361, and A. Dresden, On the iteration of linear homogeneous transformations, Bulletin of the American Math. Society, vol. 48, 1942, pp. 577–579.
7. A wealth of geometric applications of the Fourier series was given by Adolf Hurwitz in his famous paper: Sur quelques applications géométriques des séries de Fourier, Mathematische Werke, Basel, vol. 1, 1932, pp. 509–554. The above given derivation of the inequality (40) by means of the finite Fourier series is due to W. Blaschke, Kreis und Kugel, Leipzig, 1916, pp. 13–20. Blaschke avoids the use of complex numbers, a fact which robs the method of much of its elegance and requires a distinction of two cases according to the parity of $k$.

MATHEMATICAL NOTES

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ON ALMOST PRIMES*

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1. Introduction. D. H. Lehmer [1] and others have studied odd composite numbers $n$ which behave like primes in that they satisfy the congruence

$$2^n \equiv 2 \pmod{n}.$$ 

For brevity, we call such numbers almost primes. In a previous note [2] we proved that for every $k$ there exist infinitely many square free almost primes having $k$ distinct prime factors; this generalizes a result of Lehmer for $k \leq 3$. In the present note we estimate from above the number of almost primes less than a given limit.

2. Theorem. Our result is the following.

\textbf{Theorem.} Let $f(x)$ denote the number of almost primes $\leq x$. Then, for $x$ sufficiently large, we have

$$f(x) < x \exp \{ -\frac{1}{3}\log x^{1/4} \}.$$ 

\textit{Remark.} Since the number of primes $\leq x$ is asymptotic to $x/\log x$, our theorem implies that the number of almost primes $\leq x$ is very much less than the number of actual primes.

3. Proof. Let $g(n)$ be the least positive exponent $e$ such that

$$2^e \equiv 1 \pmod{n}.$$ 

We separate the almost primes $n \leq x$ into two classes $C_1$ and $C_2$. The class $C_1$ consists of those $n$'s for which

$$g(n) \leq \left[\exp (\log x^{1/3})\right] = H,$$

while $C_2$ consists of all the other almost primes $\leq x$.

The members of $C_1$ are divisors of

* Revised by D. H. Lehmer.