
Using compasses only, find the points which divide a given circle into five equal arcs.

Solution by L. S. Shively, Ball State Teachers College, Muncie, Ind.

Let $O$ be the center and $a$ be the radius of the given circle. With $a$ as radius, lay off in succession on the circle, the equal arcs $AB$, $BC$, $CD$, and $DE$. With $AC$ as radius and with $A$ and $D$ as centers, draw arcs intersecting in $F$. With $OF$ as radius and $A$ as center draw an arc intersecting the given circle in $G$. With the same radius $OF$, and $C$ and $E$ as centers, draw arcs intersecting in $H$. Then $GH$ is the radius with which, if arcs be laid off successively from any initial point on the given circle, these arcs will each be one fifth of the circumference.

For the proof, draw the lines $OA$, $OD$, $OF$, $AC$, $AF$, $AG$, $CH$, and $CE$. Let $K$ be the point of intersection of $OD$ and $CE$. The truth of each of the following statements is obvious: $A$, $O$, $D$, and $H$ are collinear; $AC=a\sqrt{3}$; $OF=AG=a\sqrt{2}$; $G$ lies on $OF$ and $OG$ is perpendicular to $OA$; $HK^2=CH^2-CK^2=2a^2-3a^2/4=5a^2/4$; and $HK=a\sqrt{5}/2$; $HO=(\sqrt{5}-1)a/2$. Hence $HO$ is the side of a regular inscribed decagon from which it follows that $GH$ is the side of a regular inscribed pentagon.


Editorial Note. The other solutions were in general slight variations of the above, and most of the solvers stated that the same type of solution is given by Mascheroni. Morley mentioned that $A$ and $G$ are consecutive vertices of the inscribed square, the construction of which is Napoleon’s problem. The proposer and Gelbart referred to two known compass constructions for bisecting, respectively, an arc of a circle and a straight line segment. The latter construction is given in the solution of 3327 [1929, 339] where four references are given to the literature of compass constructions. One of these is La Geometria del Compasso, di Lorenzo Mascheroni, Pavia, anno V della Repubblica Francese, 1797. The required construction is given in Problema 40, page 23. Also Problema 142, pp. 136–138 gives a construction for the center of a given circle. Gelbart remarks that a well known ruler and compass construction of the side of the inscribed pentagon is as follows: On the diameter $AOD$ of the given circle let $K$ be the mid-point of the radius $OD$; let $H$ be the point on $OA$ so that $KH=KG$; then $GH$ is the length of the required side. A compass construction for $G$ and $H$ is then merely a matter of applying the two above compass constructions. This he does in his solution, and concludes with a remark which in substance is as follows: Any reader of Mascheroni must be aware of the fact that any construction which is possible with ruler and compass is also possible with compasses alone, and that one should endeavor to find more beautiful solutions than the one he outlined. Bradley and La Fon state that the construction may also
be found in the second edition of the translation into French, *Géométrie du Compas*, by Carette, Paris, 1828, Problème 40, p. 50. The latter adds that it is given also on page 134 of the more recent book with the same title by A. Quenper de Lanascal, Librairie Scientifique, Albert Blanchard, Paris, 1925. Starke observed that in this problem as also in E100 [1935, 45] it is not explicitly stated that the center is given with the circle. If the circle is given without the center, the latter may be constructed by compasses alone as follows: Take a point $N$ on the given circle as center and with a radius greater than one fourth the diameter of the given circle describe a circle $(N)$ cutting the given circle in $M$ and $P$. Find the symmetric $N'$ of $N$ with respect to $MP$, and describe a circle with $N'$ as center and $N'N$ as radius cutting $(N)$ in $M'$ and $P'$. Find the symmetric $O$ of $N$ with respect to $M'P'$: then $O$ is the required center.

3710 [1934, 582]. Proposed by Harry Langman, Brooklyn, N.Y.

If the $C$'s represent binomial coefficients, show that

\[
\begin{vmatrix}
C^n_2 & C^n_3 & C^n_4 & \cdots & C^n_{n-1} & C^n_n & C^n_{n+1} \\
-(n-1) & C^n_2 & C^n_3 & \cdots & C^n_{n-2} & C^n_{n-1} & C^n_n \\
0 & -(n-2) & C^n_2 & \cdots & C^n_{n-3} & C^n_{n-2} & C^n_{n-1} \\
0 & 0 & -(n-3) & \cdots & C^n_{n-4} & C^n_{n-3} & C^n_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -2 & C^n_2 & C^n_{n+1} \\
0 & 0 & 0 & \cdots & 0 & -1 & C^n_{n+1}
\end{vmatrix} = (n!)^2.
\]

Solution by Frank Ayres, Jr., Dickinson College.

Let the given determinant be denoted by $D_n$. Then we may write

\[
D_n =
\begin{vmatrix}
C^n_2 & C^n_3 & C^n_4 & \cdots & C^n_{n-2} & C^n_{n-1} & C^n_n & C^n_{n+1} \\
-(n-1) & C^n_2 & C^n_3 & \cdots & C^n_{n-3} & C^n_{n-2} & C^n_{n-1} & C^n_n \\
0 & -(n-2) & C^n_2 & \cdots & C^n_{n-4} & C^n_{n-3} & C^n_{n-2} & C^n_{n-1} \\
0 & 0 & -(n-3) & \cdots & C^n_{n-5} & C^n_{n-4} & C^n_{n-3} & C^n_{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -2 & C^n_2 & C^n_{n+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & C^n_{n+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & C^n_{n+1} \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & C^n_{n+1} \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & C^n_{n+1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -C^n_{n+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -C^n_{n+1} \\
\end{vmatrix} = 0
\]