**Exercise**

1. In each of the following examples a scalar field is defined by the given function for all points \((x, y)\) in the plane for which the expression on the right is defined. In each example determine the set of points \((x, y)\) at which \(f\) is continuous.

   (a) \(f(x, y) = x^4 + y^4 - 4x^2y^2\)
   (b) \(f(x, y) = \ln(x^2 + y^2)\)
   (c) \(f(x, y) = \frac{1}{y} \cos x^2\)
   (d) \(f(x, y) = \tan \frac{x^2}{y}\)
   (e) \(f(x, y) = \tan^{-1} \frac{y}{x}\)
   (f) \(f(x, y) = \sin^{-1} \frac{x}{\sqrt{x^2 + y^2}}\)
   (g) \(f(x, y) = \tan^{-1} \frac{x+y}{1-xy}\)
   (h) \(f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\)
   (i) \(f(x, y) = x^4y^2\)
   (j) \(f(x, y) = \cos^{-1} \sqrt{\frac{x}{y}}\)

2. Let \(f(x, y) = \frac{x+y}{x+y}\) if \(x + y \neq 0\). Show that \(\lim_{x \to 0} [\lim_{y \to 0} f(x, y)] = 1\) but that \(\lim_{y \to 0} [\lim_{x \to 0} f(x, y)] = -1\).

3. Let \(f(x, y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}\) whenever \(x^2y^2 + (x-y)^2 \neq 0\). Show that

   \[
   \lim_{x \to 0} [\lim_{y \to 0} f(x, y)] = 0 = \lim_{y \to 0} [\lim_{x \to 0} f(x, y)]
   \]

   but that \(f(x, y)\) does not tend to a limit as \((x, y) \to (0,0)\).

**Differentiability and Gradient**

**Differentiability**

Our object here is to extend the notion of differentiability from functions of one variable to functions of several variables. Partial derivatives alone do not fulfill this role because
they reflect behavior only in particular directions. In the one-variable case we formed
the difference quotient

$$\frac{f(x + h) - f(x)}{h}$$

and called $f$ differentiable at $x$ provided that this quotient had a limit as $h$ tended to
zero. In the multivariable case we can still form the difference

$$f(x + h) - f(x)$$

but the “quotient”

$$\frac{f(x + h) - f(x)}{h}$$

makes no sense because to divide by a vector makes no sense.

We can get around this difficulty this way. If $g(h)$ is an expression in $h$, we say that
$g(h)$ is little $- o(h)$ and write $g(h) = o(h)$ if

$$\lim_{h \to 0} \frac{g(h)}{\|h\|} = 0.$$

For a function of one variable the following statements are equivalent:

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x)$$

$$\lim_{h \to 0} \frac{[f(x + h) - f(x)] - f'(x)h}{h} = 0$$

$$\lim_{h \to 0} \frac{[f(x + h) - f(x)] - f'(x)h}{\|h\|} = 0$$

$$[f(x + h) - f(x)] - f'(x)h = o(h)$$

$$[f(x + h) - f(x)] = f'(x)h + o(h).$$

Thus, for a function of one variable, the derivative of $f$ at $x$ is the unique number $f'(x)$
such that

$$f(x + h) - f(x) = f'(x)h + o(h).$$

It is this view of the derivative that inspires the notion of differentiability in the multivariable case.

Let us agree to call $o(h)$ any expression $g(h)$ for which

$$\lim_{h \to 0} \frac{g(h)}{\|h\|} = 0.$$
Definition of Differentiability

Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \). We say that \( f \) is differentiable at \( x \) if there exists a vector \( y \) such that

\[
 f(x + h) - f(x) = y \cdot h + o(h).
\]

It is not hard to show that, if such a vector \( y \) exists, it is unique. We call this unique vector the gradient of \( f \) at \( x \) and denote it by \( \nabla f(x) \):

if \( f \) is differentiable at \( x \), the gradient of \( f \) at \( x \) is the unique vector \( \nabla f(x) \) such that

\[
 f(x + h) - f(x) = \nabla f(x) \cdot h + o(h).
\]

The similarities between the one-variable case,

\[
 f(x + h) - f(x) = f'(x)h + o(h),
\]

and the multivariable case,

\[
 f(x + h) - f(x) = \nabla f(x) \cdot h + o(h),
\]

are obvious. We point to the differences. There are essentially two of them:

(1) While the derivative \( f'(x) \) is a number, the gradient \( \nabla f(x) \) is a vector.

(2) While \( f'(x)h \) is the ordinary product of two real numbers, \( \nabla f(x) \cdot h \) is the dot product of two vectors.

Calculating Gradients

We write

\[
 \nabla f(x, y) \text{ and } h = (h_1, h_2)
\]

in the two-variable case and

\[
 \nabla f(x, y, z) \text{ and } h = (h_1, h_2, h_3)
\]

for the three-variable case.

**Example 1.** For the function

\[
 f(x, y) = x^2 + y^2
\]
we have

\[ f(x + h_1, y + h_2) - f(x, y) = [(x + h_1)^2 + (y + h_2)^2] - [x^2 + y^2] \]

\[ = [2xh_1 + 2yh_2] + [h_1^2 + h_2^2] \]

\[ = (2x, 2y) \cdot h + \|h\|^2. \]

The remainder \( \|h\|^2 \) is \( o(h) \):

\[ \frac{\|h\|^2}{\|h\|} = \|h\| \to 0 \text{ as } h \to 0. \]

Thus

\[ \nabla f(x, y) = (2x, 2y). \]

**Example 2.** For the function

\[ f(x, y, z) = ax + by + cz \]

we have

\[ f(x + h_1, y + h_2, z + h_3) - f(x, y, z) \]

\[ = [a(x + h_1) + b(y + h_2) + c(z + h_3)] - [ax + by + cz] \]

\[ = ah_1 + bh_2 + ch_3 = [ai + bj + ck] \cdot h. \]

Here the remainder is identically zero and is thus certainly \( o(h) \). Therefore

\[ \nabla f(x, y, z) = (a, b, c). \]

**Theorem.** If \( f \) has continuous first partials in a neighborhood of \( x \), then \( f \) is differentiable at \( x \) and

\[ \nabla f(x, y, z) = \left( \frac{\partial f(x, y, z)}{\partial x}, \frac{\partial f(x, y, z)}{\partial y}, \frac{\partial f(x, y, z)}{\partial z} \right). \]

In two variables,

\[ \nabla f(x, y) = \left( \frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y} \right). \]

**Proof.** We prove the theorem in the two-variable case. A similar argument yields a proof in the three-variable case.
Calculating Gradients

Adding and subtracting \( f(x, y + h_2) \), we have

\[
f(x + h_1, y + h_2) - f(x, y) = [f(x + h_1, y + h_2) - f(x, y + h_2)] + [f(x, y + h_2) - f(x, y)]
\]

By the mean-value theorem for functions of one variable, we know that there are numbers

\[
0 < \theta_1 < 1 \text{ and } 0 < \theta_2 < 1
\]

such that

\[
f(x + h_1, y + h_2) - f(x, y + h_2) = \frac{\partial f}{\partial x}(x + \theta_1 h_1, y + h_2) h_1
\]

and

\[
f(x, y + h_2) - f(x, y) = \frac{\partial f}{\partial y}(x, y + \theta_2 h_2) h_2.
\]

By the continuity of \( \partial f / \partial x \),

\[
\frac{\partial f}{\partial x}(x + \theta_1 h_1, y + h_2) = \frac{\partial f}{\partial x}(x, y) + \epsilon_1(h)
\]

where

\[
\epsilon_1(h) \to 0 \text{ as } h \to 0.
\]

By the continuity of \( \partial f / \partial y \),

\[
\frac{\partial f}{\partial y}(x, y + \theta_2 h_2) = \frac{\partial f}{\partial y}(x, y) + \epsilon_2(h)
\]

where

\[
\epsilon_2(h) \to 0 \text{ as } h \to 0.
\]

Substituting these expressions, we find that

\[
f(x + h_1, y + h_2) - f(x, y) = \left[ \frac{\partial f}{\partial x}(x, y) + \epsilon_1(h) \right] h_1 + \left[ \frac{\partial f}{\partial y}(x, y) + \epsilon_2(h) \right] h_2
\]

\[
= \left( \frac{\partial f}{\partial x}(x, y) + \epsilon_1(h), \frac{\partial f}{\partial y}(x, y) + \epsilon_2(h) \right) \cdot h
\]

\[
= \left( \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right) \cdot h + \left( \epsilon_1(h), \epsilon_2(h) \right) \cdot h
\]

To complete the proof of the theorem we need only show that

\[
(\epsilon_1(h), \epsilon_2(h)) \cdot h = o(h).
\]

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From Schwarz’s inequality, $|a \cdot b| \leq \|a\| \|b\|$, we know that

$$|(\epsilon_1(h), \epsilon_2(h)) \cdot h| \leq \|(\epsilon_1(h), \epsilon_2(h))\| \|h\|.$$ 

It follows that

$$\frac{|(\epsilon_1(h), \epsilon_2(h)) \cdot h|}{\|h\|} \leq \|(\epsilon_1(h), \epsilon_2(h))\| \leq \|(\epsilon_1(h), 0)\| + \|(0, \epsilon_2(h))\|$$

$$= |\epsilon_1(h)| + |\epsilon_2(h)|,$$

by the triangle inequality. The last term tends to 0 as $h \to 0$. This completes the proof of the theorem.

**Example 3.** For

$$f(x, y) = xe^y - ye^x$$

we have

$$\frac{\partial f(x, y)}{\partial x} = e^y - ye^x, \quad \frac{\partial f(x, y)}{\partial y} = xe^y - e^x$$

and therefore

$$\nabla f(x, y) = (e^y - ye^x, xe^y - e^x).$$

When there is no reason to emphasize the point of evaluation, we don’t write $\nabla f(x, y)$ or $\nabla f(x, y, z)$ but simply $\nabla f$. Thus for the function

$$f(x, y) = xe^y - ye^x$$

we write

$$\frac{\partial f}{\partial x} = e^y - ye^x, \quad \frac{\partial f}{\partial y} = xe^y - e^x$$

and

$$\nabla f = (e^y - ye^x, xe^y - e^x).$$

**Example 4.** For

$$f(x, y, z) = \sin xy^2z^3$$

we have

$$\frac{\partial f}{\partial x} = y^2z^3 \cos xy^2z^3, \quad \frac{\partial f}{\partial y} = 2xyz^3 \cos xy^2z^3, \quad \frac{\partial f}{\partial z} = 3xy^2z^2 \cos xy^2z^3$$
Calculating Gradients

\[
\nabla f = \cos xy^2 z^3 (y^2 z^3, 2xyz, 3xy^2 z^2).
\]

**Example 5.** We take the function

\[
f(x, y, z) = x \sin \pi y + y \cos \pi z
\]

and evaluate \( \nabla f \) at \((0, 1, 2)\).

Here

\[
\frac{\partial f}{\partial x} = \sin \pi y, \quad \frac{\partial f}{\partial y} = \pi x \cos \pi y + \cos \pi z, \quad \frac{\partial f}{\partial z} = -\pi y \sin \pi z.
\]

At \((0, 1, 2)\)

\[
\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial z} = 0
\]

and thus \( \nabla f = (0, 1, 0) \).

Of special interest for later work are the powers of \( \|r\| \) where, as usual, \( r = (x, y, z) \).

We begin by showing that

\[
\nabla \|r\| = \frac{r}{\|r\|} \quad \text{and} \quad \nabla \left( \frac{1}{\|r\|} \right) = -\frac{r}{\|r\|^3}.
\]

Proof.

\[
\nabla \|r\| = \nabla (x^2 + y^2 + z^2)^{1/2}
\]

\[
= \left( \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2}, \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{1/2}, \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{1/2} \right)
\]

\[
= \left( \frac{x}{(x^2 + y^2 + z^2)^{1/2}}, \frac{y}{(x^2 + y^2 + z^2)^{1/2}}, \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \right)
\]

\[
= \frac{1}{(x^2 + y^2 + z^2)^{1/2}} (x, y, z) = \frac{r}{\|r\|},
\]

\[
\nabla \left( \frac{1}{\|r\|} \right) = \nabla \left( \frac{1}{(x^2 + y^2 + z^2)^{1/2}} \right)
\]

\[
= \left( \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-1/2}, \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-1/2}, \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-1/2} \right)
\]

\[
= \left( \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right)
\]

\[
= \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} (x, y, z) = -\frac{r}{\|r\|^3}.
\]

The formulas we just derived can be generalized: for each integer \( n \),

\[
\nabla \|r\|^n = n \|r\|^{n-2} r.
\]
Differentiability Implies Continuity

As in the one-variable case, differentiability implies continuity; namely,

if \( f \) is differentiable at \( x \), then \( f \) is continuous at \( x \).

To see this, write

\[
f(x + h) - f(x) = \nabla f(x) \cdot h + o(h)
\]

and note that

\[
|f(x + h) - f(x)| = |\nabla f(x) \cdot h + o(h)| \leq |\nabla f(x) \cdot h| + |o(h)|.
\]

As \( h \to 0 \),

\[
|\nabla f(x) \cdot h| \leq \|
abla f(x)\| \cdot \|h\| \to 0 \quad \text{and} \quad |o(h)| \to 0.
\]

It follows that

\[
[f(x + h) - f(x)] \to 0 \quad \text{and therefore} \quad f(x + h) \to f(x).
\]

Some Elementary Formulas

In many respects gradients behave just as derivatives do in the one-variable case. In particular, if \( \nabla f(x) \) and \( \nabla g(x) \) exist, then \( \nabla [f(x) + g(x)] \), \( \nabla [af(x)] \), and \( \nabla [f(x)g(x)] \) all exist, and

\[
\nabla [f(x) + g(x)] = \nabla f(x) + \nabla g(x),
\]

\[
\nabla [af(x)] = a \nabla f(x),
\]

\[
\nabla [f(x)g(x)] = \nabla f(x)g(x) + f(x) \nabla g(x).
\]

To derive the third formula, let’s assume that \( \nabla f(x) \) and \( \nabla g(x) \) both exist. Our task is to show that

\[
f(x + h)g(x + h) - f(x)g(x) = [f(x)\nabla g(x) + g(x)\nabla f(x)] \cdot h + o(h).
\]

Now

\[
f(x + h)g(x + h) - f(x)g(x)
\]

\[
= [f(x + h)g(x + h) - f(x)g(x + h)] + [f(x)g(x + h) - f(x)g(x)]
\]
Directional Derivatives

Here we take up an idea that generalizes the notion of partial derivative. Its connection to gradients will be made clear as we go on. The partial derivatives

\[ \frac{\partial}{\partial x} f(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h} \]

\[ \frac{\partial}{\partial y} f(x, y, z) = \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h} \]

\[ \frac{\partial}{\partial z} f(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h} \]

expressed in vector notation take the form

\[ \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h} \]

\[ \lim_{h \to 0} \frac{f(x + hj) - f(x)}{h} \]

\[ \lim_{h \to 0} \frac{f(x + hk) - f(x)}{h} \]

Each partial is thus the limit of a quotient

\[ \frac{f(x + hu) - f(x)}{h} \]

where \( u \) is one of the unit coordinate vectors \( i, j, \) or \( k \). There is no reason to be so restrictive on \( u \). If \( f \) is defined in a neighborhood of \( x \), then, for small \( h \), the difference quotient

\[ \frac{f(x + hu) - f(x)}{h} \]
makes sense for any unit vector $u$.

**DEFINITION** For each unit vector $u$, the limit

$$f'_u(x) = \lim_{h \to 0} \frac{f(x + hu) - f(x)}{h},$$

if it exists, is called the **directional derivative of $f$ at $x$ in the direction $u$**.

The partials are of course themselves directional derivatives:

$$f'_i(x), f'_j(x), f'_k(x).$$

As the partials of $f$ give the rates of change of $f$ in the $i, j, k$ directions, the directional derivative $f'_u$ gives the rate of change of $f$ in the direction $u$.

There is an important connection between the gradient at $x$ and the directional derivatives at $x$.

**THEOREM** If $f$ is differentiable at $x$, then $f$ has a directional derivative at $x$ in every direction $u$ and

$$f'_u(x) = \nabla f(x) \cdot u.$$

Proof. We take $u$ as a unit vector and assume that $f$ is differentiable at $x$. The differentiability at $x$ tells us that $\nabla f(x)$ exists and

$$f(x + hu) - f(x) = \nabla f(x) \cdot hu + o(hu).$$

Division by $h$ gives

$$\frac{f(x + hu) - f(x)}{h} = \nabla f(x) \cdot u + \frac{o(hu)}{h}.$$

Since

$$\left| \frac{o(hu)}{h} \right| = \frac{|o(hu)|}{|h|} = \frac{|o(hu)|}{\|hu\|} \to 0,$$

we have

$$o(hu) \to 0$$

and thus

$$\frac{f(x + hu) - f(x)}{h} \to \nabla f(x) \cdot u.$$

Earlier we saw that, if $f$ has **continuous first partials** in a neighborhood of $x$, then $f$ is differentiable at $x$ and

$$\nabla f(x) = \left( \frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(x), \frac{\partial f}{\partial z}(x) \right).$$
The next theorem shows that this formula for $\nabla f(x)$ holds wherever $f$ is differentiable.

**Theorem** If $f$ is differentiable at $x$, then all the first partials exist at $x$ and

$$\nabla f(x) = \left( \frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(x), \frac{\partial f}{\partial z}(x) \right).$$

**Proof.** Assume that $f$ is differentiable at $x$. Then $\nabla f(x)$ exists and we can write

$$\nabla f(x) = [\nabla f(x) \cdot i]i + [\nabla f(x) \cdot j]j + [\nabla f(x) \cdot k]k.$$ 

The result follows from observing that

$$\nabla f(x) \cdot i = f_i'(x) = \frac{\partial f}{\partial x}(x),$$

$$\nabla f(x) \cdot j = f_j'(x) = \frac{\partial f}{\partial y}(x),$$

$$\nabla f(x) \cdot k = f_k'(x) = \frac{\partial f}{\partial z}(x).$$

**Problem 1.** Find the directional derivative of the function

$$f(x, y) = x^2 + y^2$$

at the point $(1, 2)$ in the direction of the vector $2i - 3j$.

**Solution.** In the first place, $2i - 3j$ is not a unit vector; its norm is $\sqrt{13}$. The unit vector in the direction of $2i - 3j$ is the vector

$$u = \frac{(2, 3)}{\sqrt{13}}.$$ 

Next

$$\nabla f = 2xi + 2yj,$$

and therefore

$$\nabla f(1, 2) = 2i + 4j.$$ 

Hence

$$f_u'(1, 2) = \nabla f(1, 2) \cdot u = (2, 4) \cdot \frac{(2, 3)}{\sqrt{13}} = -\frac{8}{\sqrt{13}}.$$ 

**Problem 2.** Find the directional derivative of the function

$$f(x, y, z) = x \cos y \sin z$$

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at the point \((1, \pi, \pi/4)\) in the direction of the vector \(2i - j + 4k\).

Solution. The unit vector in the direction of \(2i - j + 4k\) is the vector

\[
u = \frac{1}{\sqrt{21}}[2i - j + 4k].
\]

Here

\[
\frac{\partial f}{\partial x} = \cos y \sin z, \quad \frac{\partial f}{\partial y} = -x \sin y \sin z, \quad \frac{\partial f}{\partial z} = x \cos y \cos z
\]

so that

\[
\frac{\partial f}{\partial x}(1, \pi, \pi/4) = -\frac{1}{\sqrt{2}}, \quad \frac{\partial f}{\partial y}(1, \pi, \pi/4) = 0, \quad \frac{\partial f}{\partial z}(1, \pi, \pi/4) = -\frac{1}{\sqrt{2}}.
\]

Therefore

\[
\nabla f(1, \pi, \pi/4) = -\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} k
\]

\[
f_u'(1, \pi, \pi/4) = \nabla f(1, \pi, \pi/4) \cdot u
\]

\[
= -\frac{1}{\sqrt{2}} [i + k] \cdot \frac{1}{\sqrt{21}}[2i - j + 4k] = -\frac{\sqrt{42}}{7}.
\]

Note that the directional derivative in a direction \(u\) is the component of the gradient vector in that direction.

If \(\nabla f(x) \neq 0\), then

\[
f_u'(x) = \nabla f(x) \cdot u = \|\nabla f(x)\| \|u\| \cos \theta = \|\nabla f(x)\| \cos \theta
\]

where \(\theta\) is the angle between \(\nabla f(x)\) and \(u\). It follows from the Cauchy-Schwarz’s inequality that

\[-\|\nabla f(x)\| \leq f_u'(x) \leq \|\nabla f(x)\|
\]

for all directions \(u\). If \(u\) points in the direction of \(\nabla f(x)\), then

\[
f_u'(x) = \|\nabla f(x)\|
\]

and, if \(u\) points in the direction of \(-\nabla f(x)\)

\[
f_u'(x) = -\|\nabla f(x)\|.
\]

Since the directional derivative gives the rate of change of the function in that direction, it is clear that
a differentiable function $f$ increases most rapidly in the direction of the gradient (the rate of change is then $\|\nabla f(x)\|$ and it decreases most rapidly in the opposite direction (the rate of change is then $-\|\nabla f(x)\|$).

**Example 3.** The graph of the function

$$f(x, y) = \sqrt{1 - x^2 - y^2}$$

is the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$. The function is defined on the closed unit disc but differentiable only on the open unit disc.

The gradient

$$\nabla f(x, y) = \left(\frac{-x}{\sqrt{1 - x^2 - y^2}}, \frac{-y}{\sqrt{1 - x^2 - y^2}}\right)$$

is a negative multiple of $r$:

$$\nabla f(x, y) = \frac{-1}{\sqrt{1 - x^2 - y^2}}(x, y) = \frac{-1}{\sqrt{1 - x^2 - y^2}}r.$$ 

Since $r$ points from the origin to $(x, y)$, the gradient points from $(x, y)$ to the origin. This means that $f$ increases most rapidly toward the origin. This is borne out by the observation that along the hemispherical surface the path of steepest ascent from the point $P(x, y, f(x, y))$ is the “great circle route to the north pole.”

**Problem 4.** The temperature at each point of a metal plate is given by the function

$$T(x, y) = e^x \cos y + e^y \cos x.$$ 

(a) In what direction does the temperature increase most rapidly at the point $(0, 0)$? What is this rate of increase?

(b) In what direction does the temperature decrease most rapidly at $(0, 0)$?

Solution

$$\nabla T(x, y) = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right) = (e^x \cos y - e^y \sin x, e^y \cos x - e^x \sin y).$$

(a) At $(0, 0)$ the temperature increases most rapidly in the direction of the gradient

$$\nabla T(0, 0) = i + j.$$ 

This rate of increase is

$$\|\nabla T(0, 0)\| = \|i + j\| = \sqrt{2}.$$ 

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(b) The temperature decreases most rapidly in the direction of

\[-\nabla T(0,0) = -i - j.\]

**Problem 5.** The mass density (mass per unit volume) of a metal ball centered at the origin is given by the function

\[\lambda(x, y, z) = ke^{-(x^2+y^2+z^2)},\]

\[k\] a positive constant.

(a) In what direction does the density increase most rapidly at the point \((x, y, z)\)?

What is this rate of density increase?

(b) In what direction does the density decrease most rapidly?

(c) What are the rates of density change at \((x, y, z)\) in the \(i, j, k\) directions?

**Solution.** The gradient

\[\nabla \lambda(x, y, z) = \left(\frac{\partial \lambda}{\partial x}, \frac{\partial \lambda}{\partial y}, \frac{\partial \lambda}{\partial z}\right) = -2ke^{-(x^2+y^2+z^2)}(x, y, z)\]

\[= -2\lambda(x, y, z)r\]

points from \((x, y, z)\) in the direction opposite to that of the radius vector.

(a) The density increases most rapidly toward the origin. The rate of increase is

\[\|\nabla \lambda(x, y, z)\| = 2\lambda(x, y, z)\|r\| = 2\lambda(x, y, z)\sqrt{x^2 + y^2 + z^2}\]

(b) The density decreases most rapidly directly away from the origin.

(c) The rates of density change in the \(i, j, k\) directions are given by the directional derivatives

\[\lambda'_i(x, y, z) = \nabla \lambda(x, y, z) \cdot i = -2x\lambda(x, y, z),\]

\[\lambda'_j(x, y, z) = \nabla \lambda(x, y, z) \cdot j = -2y\lambda(x, y, z),\]

\[\lambda'_k(x, y, z) = \nabla \lambda(x, y, z) \cdot k = -2z\lambda(x, y, z).\]

These are just the first partials of \(\lambda\).

**Problem 6.** The temperature at each point of a metal plate is given by the function

\[T(x, y) = A + x^2 - y^2.\]
Find the path followed by a heat-seeking particle that originates at \((-2, 1)\).

Solution. The particle moves in the direction of the gradient vector

$$\nabla T(x, y) = 2xi - 2yj.$$  

We want the curve

$$C : r(t) = x(t)i + y(t)j$$

which begins at \((-2, 1)\) and at each point has its tangent vector in the direction of \(\nabla T\).

We can satisfy the first condition by setting

$$x(0) = -2, y(0) = 1.$$  

We can satisfy the second condition by setting

$$x'(t) = 2x(t), y'(t) = -2y(t).$$

These differential equations, together with initial conditions at \(t = 0\), imply that

$$x(t) = -2e^{2t}, y(t) = e^{-2t}.$$  

We can eliminate the parameter \(t\) by noting that

$$x(t)y(t) = (-2e^{2t})(e^{-2t}) = -2.$$  

In terms of just \(x\) and \(y\) we have

$$xy = -2.$$  

The particle moves from the point \((-2, 1)\) along the left branch of the hyperbola \(xy = -2\) in the direction of decreasing \(x\).

Remark. The pair of differential equations

$$x'(t) = 2x(t), y'(t) = -2y(t)$$

can be set as a single differential equation in \(x\) and \(y\): the relation

$$\frac{y'(t)}{x'(t)} = -\frac{y(t)}{x(t)}$$

gives

$$\frac{dy}{dx} = -\frac{y}{x}.$$
This equation is readily solved directly:

\[
\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x}
\]

\[
\ln |y| = -\ln |x| + C
\]

\[
\ln |y| + \ln |x| = C
\]

\[
\ln |xy| = C
\]

Thus \( xy \) is constant:

\[
xy = k.
\]

Since the curve passes through the point \((-2, 1), k = -2\) and once again we have the curve

\[
xy = -2.
\]