THE VECTOR DIFFERENTIAL OPERATOR $\nabla$

Divergence $\nabla \cdot v$, Curl $\nabla \times v$

The vector differential operator $\nabla$ is defined by setting

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k.$$ 

This is no ordinary vector. Its “components” are differentiation symbols. As the term “operator” suggests, $\nabla$ is to be thought of as something that “operates” on things. What sort of things? Scalar fields and vector fields.

Suppose that $f$ is a differentiable scalar field. Then $\nabla$ operates on $f$ as follows:

$$\nabla f = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$ 

This is just the gradient of $f$, with which we are already familiar.

How does $\nabla$ operate on vector fields? In two ways. If $v = v_1 i + v_2 j + v_3 k$ is a differentiable vector field, then by definition

$$\nabla \cdot v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

and

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) i + \left( \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) j + \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) k.$$ 

The first “product,” $\nabla \cdot v$, defined in imitation of the ordinary dot product, is called the divergence of $v$:

$$\nabla \cdot v = \text{div } v.$$ 

The second “product,” $\nabla \times v$, defined in imitation of the ordinary cross product, is called the curl of $v$:

$$\nabla \times v = \text{curl } v.$$ 

Interpretation of Divergence and Curl

Suppose we know the divergence of a field and also the curl. What does that tell us? For definitive answers we must wait for the divergence theorem and Stokes’s theorem, but, in a preliminary way, we can give some rough answers right now. View $v$ as the velocity field of some fluid. The divergence of $v$ at a point $P$ gives us an indication of
whether the fluid tends to accumulate near $P$ (negative divergence) or tends to move away from $P$ (positive divergence). The curl at $P$ measures the rotational tendency of the fluid.

**Example 1.** Set

$$v(x, y, z) = \alpha x\hat{i} + \alpha y\hat{j} + \alpha z\hat{k}.$$ (\(\alpha\) a constant)

Divergence:

$$\nabla \cdot v = \alpha \frac{\partial x}{\partial x} + \alpha \frac{\partial y}{\partial y} + \alpha \frac{\partial z}{\partial z} = 3\alpha.$$ 

Curl:

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x & \alpha y & \alpha z \end{vmatrix} = 0$$

because the partial derivatives that appear in the expanded determinant are all zero.

The field of Example 1

$$v(x, y, z) = \alpha x\hat{i} + \alpha y\hat{j} + \alpha z\hat{k} = r$$

can be viewed as the velocity field of a fluid in radial motion–toward the origin if $\alpha < 0$, away from the origin if $\alpha > 0$. Consider a point $(x, y, z)$, a spherical neighborhood of that point, and a cone emanating from the origin that is tangent to the boundary of the neighborhood.

Note two things: all the fluid in the cone stays in the cone, and the speed of the fluid is proportional to its distance from the origin. Therefore, if the divergence $3\alpha$ is negative, then $\alpha$ is negative, the motion is toward the origin, and the neighborhood gains fluid because the fluid coming in is moving more quickly than the fluid going out. (Also the entry area is greater than the exit area.) If, however, the divergence $3\alpha$ is positive, then $\alpha$ is positive, the motion is away from the origin, and the neighborhood loses fluid because the fluid coming in is moving more slowly than the fluid going out. (Also the entry area is smaller than the exit area.)

Since the motion is radial, the fluid has no rotational tendency whatsoever, and we would expect the curl to be identically zero. It is.

**Example 2.** Set

$$v(x, y, z) = -\omega y\hat{i} + \omega x\hat{j},$$ (\(\omega\) a positive constant).

Divergence:

$$\nabla \cdot v = -\omega \frac{\partial y}{\partial x} + \omega \frac{\partial x}{\partial y} = 0 + 0 = 0.$$
Curl:
\[ \nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \{\omega \frac{\partial x}{\partial y} + \omega \frac{\partial y}{\partial x}\}k = 2\omega k. \]

The field of Example 2
\[ v(x, y, z) = -\omega yi + \omega xj, \text{ (\omega a positive constant)}. \]
is the velocity field of uniform counterclockwise rotation about the \( z \)-axis with angular speed \( \omega \). We can see this by noting that \( v \) is perpendicular to \( r \):
\[ v \cdot r = (-\omega yi + \omega xj) \cdot (xi + yj + zk) = -\omega yx + \omega xy = 0. \]

and the speed at each point is \( \omega R \) where \( R \) is the radius of rotation:
\[ v = \sqrt{\omega^2 y^2 + \omega^2 x^2} = \omega \sqrt{y^2 + x^2} = \omega R. \]

How is the curl, \( 2\omega k \), related to the rotation? The angular velocity vector is the vector \( \omega k \). In this case then the curl of \( v \) is twice the angular velocity vector.

With this rotation no neighborhood gains any fluid and no neighborhood loses any fluid. As we saw, the divergence is identically zero.

**Basic Identities**

For vectors we have \( a \times a = 0 \). Is it true that \( \nabla \times \nabla = 0 \)? We make it true by defining
\[ (\nabla \times \nabla)f = \nabla \times (\nabla f). \]

**THEOREM (THE CURL OF A GRADIENT IS ZERO)**

If \( f \) is a scalar field with continuous second partials, then
\[ (\nabla \times \nabla)f = 0. \]

**Proof**
\[ (\nabla \times \nabla)f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = 0, \]
by the equality of the mixed partials.

For vectors we have \( a \cdot (a \times c) = 0 \). The analogous operator formula, \( \nabla \cdot (\nabla \times v) = 0 \), is also valid.
**THEOREM** (THE DIVERGENCE OF A CURL IS ZERO)

If the components of the vector field \( v = v_1 i + v_2 j + v_3 k \) have continuous second partials, then

\[
\nabla \cdot (\nabla \times v) = 0.
\]

Proof. Again the key is the equality of the mixed partials:

\[
\nabla \cdot (\nabla \times v) = 0 = \frac{\partial}{\partial x} \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) = 0
\]

since for each component the mixed partials cancel.

The next two identities are product rules. Here \( f \) is a scalar field and \( v \) is a vector field.

\[
\nabla \cdot (fv) = (\nabla f) \cdot v + f(\nabla \cdot v). \quad [\text{div } (fv) = (\text{grad } f) \cdot v + f(\text{div } v)]
\]

\[
\nabla \times (fv) = (\nabla f) \times v + f(\nabla \times v). \quad [\text{curl } (fv) = (\text{grad } f) \times v + f(\text{curl } v)]
\]

We know from Example 1 that \( \nabla \cdot r = 3 \) and \( \nabla \times r = 0 \) at all points of space. Now we can show that, if \( n \) is an integer, then, for all \( r \neq 0 \),

\[
\nabla \cdot (r^n r) = (n + 3)r^n \text{ and } \nabla \times (r^n r) = 0.
\]

Proof. Recall that \( \nabla r^n = nr^{n-2} \cdot r \). We have

\[
\nabla \cdot (r^n r) = (\nabla r^n) \cdot r + r^n (\nabla \cdot r)
\]

\[
= (nr^{n-2} \cdot r + r^n (3)) = nr^{n-2} (r \cdot r) + 3r^n = (n + 3)r^n.
\]

Since

\[
r^n r = \nabla \left( r^{n+2} \right) / (n + 2),
\]

it follows that

\[
\nabla \times (r^n r) = 0.
\]

**The Laplacean**

From the operator \( \nabla \) we can construct other operators. The most important of these is the Laplacean

\[
\nabla^2 = \nabla \cdot \nabla.
\]

The Laplacean (named after the French mathematician Pierre-Simon Laplace) operates on scalar fields according to the following rule:

\[
\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\]
**Example 3.** If \( f(x, y, z) = x^2 + y^2 + z^2 \), then
\[
\nabla^2 f = \frac{\partial^2}{\partial x^2} (x^2 + y^2 + z^2) + \frac{\partial^2}{\partial y^2} (x^2 + y^2 + z^2) + \frac{\partial^2}{\partial z^2} (x^2 + y^2 + z^2)
\]
\[
= 2 + 2 + 2 = 6.
\]

**Example 4.** If \( f(x, y, z) = e^{xyz} \), then
\[
\nabla^2 f = \frac{\partial^2}{\partial x^2} (e^{xyz}) + \frac{\partial^2}{\partial y^2} (e^{xyz}) + \frac{\partial^2}{\partial z^2} (e^{xyz}) = (y^2z^2 + x^2z^2 + x^2y^2)e^{xyz}.
\]

**Example 5.** To calculate \( \nabla^2 (\sin r) = \nabla^2 (\sin \sqrt{x^2 + y^2 + z^2}) \) we could write
\[
\frac{\partial^2}{\partial x^2} \left( \sin \sqrt{x^2 + y^2 + z^2} \right) + \frac{\partial^2}{\partial y^2} \left( \sin \sqrt{x^2 + y^2 + z^2} \right) + \frac{\partial^2}{\partial z^2} \left( \sin \sqrt{x^2 + y^2 + z^2} \right)
\]
and proceed from there. The calculations are straightforward but lengthy. We will do it in a different way.

Recall that
\[
\nabla^2 f = \nabla \cdot \nabla f, \quad \nabla \cdot (fv) = (\nabla f) \cdot v + f(\nabla \cdot v)
\]
\[
\nabla f(r) = f'(r)r^{-1}r, \quad \nabla \cdot (r^n r) = (n + 3)r^n
\]
Using these relations, we have
\[
\nabla^2 (\sin r) = \nabla \cdot \nabla \sin r = \nabla \cdot [(\cos r)r^{-1}r]
\]
\[
= [(\nabla \cos r) \cdot r^{-1}r] + \cos r(\nabla \cdot r^{-1}r)
\]
\[
= \{[(\sin r)r^{-1}r] \cdot r^{-1}r\} + \cos r(2r^{-1})
\]
\[
= \sin r + 2r^{-1} \cos r.
\]

**THE DIVERGENCE THEOREM**

Green’s theorem enables us to express a double integral over a Jordan region \( \Omega \) with a piecewise-smooth boundary \( C \) as a line integral over \( C \):
\[
\iint_{\Omega} \left[ \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) \right] \, dxdy = \oint_C P(x, y) \, dx + Q(x, y) \, dy.
\]
In vector terms this relation can be written
\[
\iint_{\Omega} (\nabla \cdot v) \, dxdy = \int_C (v \cdot n)ds.
\]
Here $n$ is the outer unit normal and the integral on the right is taken with respect to arc length.

Proof. Set $v = Qi - Pj$. Then

$$\iint_\Omega (\nabla \cdot v) \, dx dy = \iint_\Omega \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, dx dy.$$ 

All we have to show then is that

$$\int_C (v \cdot n) \, ds = \oint_C P(x, y) \, dx + Q(x, y) \, dy.$$ 

For $C$ traversed counterclockwise, $n = T \times k$, where $T$ is the unit tangent vector. Thus

$$v \cdot n = v \cdot (T \times k) = (-v) \cdot (k \times T) = (-v \times k) \cdot T.$$ 

Since $-v \times k = (Pj - Qi) \times k = Pi + Qj$, we have $v \cdot n = (Pi + Qj) \cdot T$. Therefore

$$\int_C (v \cdot n) \, ds = \int_C [(Pi + Qj) \cdot T] \, ds$$

$$= \int_C (Pi + Qj) \cdot dr = \oint_C P(x, y) \, dx + Q(x, y) \, dy.$$ 

Green’s theorem expressed has a higher dimensional analog that is known as the divergence theorem.

**THEOREM** THE DIVERGENCE THEOREM

Let $T$ be a solid bounded by a closed surface $S$ which, if not smooth, is piecewise smooth. If the vector field $v = v(x, y, z)$ is continuously differentiable throughout $T$, then

$$\iiint_T (\nabla \cdot v) \, dx dy dz = \iint_S (v \cdot n) \, d\sigma$$

where $n$ is the outer unit normal.

Proof. We will carry out the proof under the assumption that $S$ is smooth and that any line parallel to a coordinate axis intersects $S$ at most twice. Our first step is to express the outer unit normal $n$ in terms of its direction cosines:

$$n = \cos \alpha_1 i + \cos \alpha_2 j + \cos \alpha_3 k.$$ 

Then

$$v \cdot n = v_1 \cos \alpha_1 + v_2 \cos \alpha_2 + v_3 \cos \alpha_3.$$ 

The idea of the proof is to show that

$$\int_S v_1 \cos \alpha_1 \, d\sigma = \iiint_T \frac{\partial v_1}{\partial x} \, dx dy dz,$$
\[ \iint_S v_2 \cos \alpha_2 \, d\sigma = \iiint_T \frac{\partial v_2}{\partial y} \, dxdydz, \]
\[ \iint_S v_3 \cos \alpha_3 \, d\sigma = \iiint_T \frac{\partial v_3}{\partial z} \, dxdydz. \]

All three equations can be verified in much the same manner. We will carry out the details only for the third equation.

Let \( \Omega_{xy} \) be the projection of \( T \) onto the \( xy \)-plane. If \((x, y) \in \Omega_{xy}\) then, by assumption, the vertical line through \((x, y)\) intersects \( S \) in at most two points, an upper point \( P^+ \) and a lower point \( P^- \). (If the vertical line intersects \( S \) at only one point \( P \), we set \( P = P^+ = P^- \).) As \((x, y)\) ranges over \( \Omega_{xy} \), the upper point \( P^+ \) describes a surface

\[ S^+ : z = f^+(x, y), (x, y) \in \Omega_{xy} \]

and the lower point describes a surface

\[ S^- : z = f^-(x, y), (x, y) \in \Omega_{xy}. \]

By our assumptions, \( f^+ \) and \( f^- \) are continuously differentiable, \( S = S^+ \cup S^- \), and the solid \( T \) is the set of all points \((x, y, z)\) with

\[ f^-(x, y) \leq z \leq f^+(x, y), (x, y) \in \Omega_{xy}. \]

Now let \( \gamma \) be the angle between the positive \( z \)-axis and the upper unit normal. On \( S^+ \) the outer unit normal \( n \) is the upper unit normal. Thus on \( S^+ \)

\[ \gamma = \alpha_3 \text{ and } \cos \alpha_3 \sec \gamma = 1. \]

On \( S^- \) the outer unit normal \( n \) is the lower unit normal. In this case

\[ \gamma = \pi - \alpha_3 \text{ and } \cos \alpha_3 \sec \gamma = -1. \]

Thus,

\[ \iint_{S^+} v_3 \cos \alpha_3 \, d\sigma = \iint_{\Omega_{xy}} v_3 \cos \alpha_3 \sec \gamma \, dxdy = \iint_{\Omega_{xy}} v_3[x, y, f^+(x, y)] \, dxdy \]

and

\[ \iint_{S^-} v_3 \cos \alpha_3 \, d\sigma = \iint_{\Omega_{xy}} v_3 \cos \alpha_3 \sec \gamma \, dxdy = -\iint_{\Omega_{xy}} v_3[x, y, f^-(x, y)] \, dxdy. \]

It follows that

\[ \iint_S v_3 \cos \alpha_3 \, d\sigma = \iint_{S^+} v_3 \cos \alpha_3 \, d\sigma + \iint_{S^-} v_3 \cos \alpha_3 \, d\sigma \]

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\[
\begin{align*}
&= \iint_{\Omega_{xy}} \left( v_3[x, y, f^+(x, y)] - v_3[x, y, f^-(x, y)] \right) \, dx \, dy \\
&= \iint_{\Omega_{xy}} \left( \int_{f^-(x, y)}^{f^+(x, y)} \frac{\partial v_3}{\partial z}(x, y, z) \, dz \right) \, dx \, dy \\
&= \iiint_{\mathcal{T}} \frac{\partial v_3}{\partial z}(x, y, z) \, dx \, dy \, dz.
\end{align*}
\]

This confirms the third equation. The second equation can be confirmed by projection onto the \(xz\)-plane; the first equation can be confirmed by projection onto the \(yz\)-plane.

**Divergence as Outward Flux per Unit Volume**

Choose a point \(P\) and surround it by a closed ball \(N_\epsilon\), of radius \(\epsilon\). According to the divergence theorem

\[
\iiint_{N_\epsilon} (\nabla \cdot v) \, dx \, dy \, dz = \text{flux of } v \text{ out of } N_\epsilon.
\]

Thus

\[
(\text{average divergence of } v \text{ on } N_\epsilon) \times (\text{volume of } N_\epsilon) = \text{flux of } v \text{ out of } N_\epsilon
\]

and

\[
\text{average divergence of } v \text{ on } N_\epsilon = \frac{\text{flux of } v \text{ out of } N_\epsilon}{\text{volume of } N_\epsilon}.
\]

Taking the limit of both sides as \(\epsilon\) shrinks to 0, we have

\[
\text{divergence of } v \text{ at } P = \lim_{\epsilon \to 0^+} \frac{\text{flux of } v \text{ out of } N_\epsilon}{\text{volume of } N_\epsilon}.
\]

In this sense **divergence is outward flux per unit volume.**

Think of \(v\) as the velocity of a fluid. Negative divergence at \(P\) signals an accumulation of fluid near \(P\):

\[
\nabla \cdot v < 0 \text{ at } P \Rightarrow \text{flux out of } N_\epsilon < 0 \Rightarrow \text{net flow into } N_\epsilon.
\]

Positive divergence at \(P\) signals a flow of liquid away from \(P\):

\[
\nabla \cdot v > 0 \text{ at } P \Rightarrow \text{flux out of } N_\epsilon > 0 \Rightarrow \text{net flow out of } N_\epsilon.
\]

Points at which the divergence is negative are called **sinks**; points at which the divergence is positive are called **sources**. If the divergence of \(v\) is 0 throughout, then the flow has no sinks and no sources and \(v\) is called **solenoidal.**
The divergence theorem, stated for solids bounded by a single closed surface, can be extended to solids bounded by several closed surfaces. Suppose, for example, that we start with a solid bounded by a closed surface $S_1$ and extract from the interior of that solid a solid bounded by a closed surface $S_2$. The remaining solid, call it $T$ has a boundary $S$ that consists of two pieces: an outer piece $S_1$ and an inner piece $S_2$. The key here is to note that the outer normal for $T$ points out of $S_1$ but into $S_2$. The divergence theorem can be proven for $T$ by slicing $T$ into two pieces $T_1$ and $T_2$ and applying the divergence theorem to each piece:

$$\iiint_{T_1} (\nabla \cdot v) \, dx dy dz = \iint_{\text{boundary of } T_1} (v \cdot n) \, d\sigma$$

$$\iiint_{T_2} (\nabla \cdot v) \, dx dy dz = \iint_{\text{boundary of } T_2} (v \cdot n) \, d\sigma$$

The triple integrals over $T_1$ and $T_2$ add up to the triple integral over $T$. When the surface integrals are added together, the integrals along the common cut cancel (because the normals are in opposite directions) and therefore only the integrals over $S_1$ and $S_2$ remain. Thus the surface integrals add up to the surface integral over $S : S_1 \cup S_2$ and the divergence theorem still holds:

$$\iiint_T (\nabla \cdot v) \, dx dy dz = \iint_S (v \cdot n) \, d\sigma.$$

**An Application to Static Charges**

Consider a point charge $q$ somewhere in space. This charge creates around itself an electric field $E$, which in turn exerts an electric force on every other nearby charge. If we center our coordinate system at $q$, then the electric field at $r$ can be written

$$E(r) = q \frac{r}{r^3}.$$ 

(This is found experimentally.) Note that this field has exactly the same form as the gravitational field: a constant multiple of $r^{-3}r$. It follows from the formula

$$\nabla \cdot (r^n r) = (n + 3)r^n$$

that

$$\nabla \cdot E = 0 \text{ for all } r \neq 0.$$
We are interested in the flux of $\mathbf{E}$ out of a closed surface $S$. We assume that $S$ does not pass through $q$.

If the charge $q$ is outside of $S$, then $\mathbf{E}$ is continuously differentiable on the region $T$ bounded by $S$, and, by the divergence theorem,

$$\text{flux of } \mathbf{E} \text{ out of } S = \iiint_T (\nabla \cdot \mathbf{E}) \, dx\,dy\,dz = \iiint_T 0 \, dx\,dy\,dz = 0.$$ 

If $q$ is inside of $S$, then the divergence theorem does not apply to $T$ directly because $\mathbf{E}$ is not differentiable on all of $T$. We can circumvent this difficulty by surrounding $q$ by a small sphere $S_a$ of radius $a$ and applying the divergence theorem to the region $T'$ bounded on the outside by $S$ and on the inside by $S_a$.

Since $\mathbf{E}$ is continuously differentiable on $T'$,

$$\iiint_{T'} (\nabla \cdot \mathbf{E}) \, dx\,dy\,dz = \text{flux of } \mathbf{E} \text{ out of } S + \text{flux of } \mathbf{E} \text{ into } S_a$$

$$= \text{flux of } \mathbf{E} \text{ out of } S - \text{flux of } \mathbf{E} \text{ out of } S_a.$$

Since $\nabla \cdot \mathbf{E} = 0$ on $T'$, the triple integral on the left is zero and therefore

$$\text{flux of } \mathbf{E} \text{ out of } S = \text{flux of } \mathbf{E} \text{ out of } S_a.$$

The quantity on the right is easy to calculate: on $S_a$, $\mathbf{n} = \mathbf{r}/r$ and therefore

$$E \cdot \mathbf{n} = q \frac{\mathbf{r}}{r^3} \cdot \frac{\mathbf{r}}{r} = \frac{q}{r^2} = \frac{q}{a^2}.$$

Thus

$$\text{flux of } \mathbf{E} \text{ out of } S_a = \iint_{S_a} (\mathbf{E} \cdot \mathbf{n}) \, d\sigma = \iint_{S_a} \frac{q}{a^2} \, d\sigma = \frac{q}{a^2} \text{(area of } S_a) = 4\pi q.$$

It follows that the flux of $\mathbf{E}$ out of $S = 4\pi q$.

In summary, $\mathbf{E}$ is the electric field of a point charge $q$ and $S$ is a closed surface that does not pass through $q$, then

$$\text{flux of } \mathbf{E} \text{ out of } S = \begin{cases} 0, & \text{if } q \text{ is outside of } S \\ 4\pi q, & \text{if } q \text{ is inside } S \end{cases}.$$