REDUCTION TO REPEATED INTEGRALS

Let $T$ be a solid in $\mathbb{R}^3$. The projection of $T$ onto the $xy$-plane is denoted by $\Omega_{xy}$. The solid $T$ is then the set of all $(x, y, z)$ with

$$(x, y) \in \Omega_{xy} \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y).$$

The triple integral over $T$ can be evaluated by setting

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iint_{\Omega_{xy}} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) \, dz \right) \, dx \, dy.$$

Suppose that $\Omega_{xy}$ takes the form $a_1 \leq x \leq a_2, \phi_1(x) \leq y \leq \phi_2(x)$. Then $T$ itself is the set of all $(x, y, z)$ with $a_1 \leq x \leq a_2, \phi_1(x) \leq y \leq \phi_2(x), \psi_1(x, y) \leq z \leq \psi_2(x, y)$.

The triple integral over $T$ can then be expressed by three ordinary integrals:

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \int_{a_1}^{a_2} \left[ \int_{\phi_1(x)}^{\phi_2(x)} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) \, dz \right) \, dy \right] \, dx.$$

It is customary to omit the brackets and parentheses and write

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

Here we first integrate with respect to $z$ [from $z = \psi_1(x, y)$ to $z = \psi_2(x, y)$], then with respect to $y$ [from $y = \phi_1(x)$ to $y = \phi_2(x)$], and finally with respect to $x$ [from $x = a_1$ to $x = a_2$].

There is nothing sacred about this order of integration. Other orders of integration are possible and in some cases more convenient. Suppose, for example, that the projection of $T$ onto the $xz$-plane is a region of the form

$$\Omega_{xz} : a_1 \leq z \leq a_2, \phi_1(z) \leq x \leq \phi_2(z).$$

If $T$ is the set of all $(x, y, z)$ with $a_1 \leq z \leq a_2, \phi_1(z) \leq x \leq \phi_2(z), \psi_1(x, z) \leq y \leq \psi_2(x, z)$.

then

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \int_{a_1}^{a_2} \int_{\phi_1(z)}^{\phi_2(z)} \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x, y, z) \, dy \, dx \, dz.$$

In this case we integrate first with respect to $y$, then with respect to $x$, and finally with respect to $z$. Still four other orders of integration are possible.

**Problem 1.** Use triple integration to find the volume of the tetrahedron $T$ bounded by the planes

$$x = 0, y = 0, z = 0, x + y + z = 1.$$
Where is the centroid?

Solution. The volume of $T$ is given by the triple integral

$$V = \iiint_T dxdydz.$$  

To evaluate this triple integral we can project $T$ onto any one of the three coordinate planes. We will project onto the $xy$-plane. The base region is then the triangle

$$\Omega_{xy} : 0 \leq x \leq 1, 0 \leq y \leq 1 - x.$$ 

Since the inclined face is part of the plane $z = 1 - x - y$, we have $T$ as the set of all $$(x, y, z)$$ with

$$0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y.$$ 

It follows that

$$V = \iiint_T dxdydz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz
dy
dx = \int_0^1 \int_0^{1-x} (1 - x - y)
dy
dx = \int_0^1 \int_0^{1-x} \frac{1}{2}(1 - x)^2
dx = \frac{1}{6}.$$ 

By symmetry $\bar{x} = \bar{y} = \bar{z}$. We can calculate $\bar{x}$ as follows:

$$\bar{x}V = \iiint_T x dxdydz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xdz
dy
dx = \frac{1}{24}.$$ 

Since $V = \frac{1}{6}$ we have $\bar{x} = \frac{1}{4}$. The centroid is the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

Problem 2. Find the mass of a solid right circular cylinder of radius $r$ and height $h$ given that the mass density varies directly with the distance from one of the bases.

Solution. Call the solid $T$. We can characterize $T$ by the following inequalities:

$$-r \leq x \leq r, -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2}, 0 \leq z \leq h.$$ 

The first two inequalities define the base region $\Omega_{xy}$. Supposing that the density varies directly with the distance from the lower base, we have $\lambda(x, y, z) = kz$. Then

$$M = \iiint_T kz
dxdydz = \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^h kz
dz
dy
dx = \frac{1}{2}kh^2 \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy
dx = \frac{1}{2}kh^2 r^2 \pi.$$ 

Remark. In Problem 2 we would have profited by not skipping the double integral stage; namely, we could have written

$$M = \int_{\Omega_{xy}} \left(\int_0^h kz
dz\right) dxdy = \int_{\Omega_{xy}} \frac{1}{2}kh^2
dxdy = \frac{1}{2}kh^2 (\text{area of } \Omega_{xy}).$$
Problem 3. Integrate \( f(x, y, z) = yz \) over that part of the first octant \( x \geq 0, y \geq 0, z \geq 0 \) that is cut off by the ellipsoid
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
\]

Solution. Call the solid \( T \). The upper boundary of \( T \) has equation
\[
z = \psi(x, y) = \frac{c}{ab} \sqrt{a^2 b^2 - b^2 x^2 - a^2 y^2}.
\]
This surface intersects the \( xy \)-plane in the curve
\[
y = \phi(x) = \frac{b}{a} \sqrt{a^2 - x^2}.
\]
We can take
\[
\Omega_{xy} : 0 \leq x \leq a, 0 \leq y \leq \frac{b}{a} \sqrt{a^2-x^2}.
\]
as the base region and characterize \( T \) as the set of all \((x, y, z)\) with
\[
0 \leq x \leq a, 0 \leq y \leq \phi(x), 0 \leq z \leq \psi(x, y).
\]
We can therefore calculate the triple integral by evaluating
\[
\int_0^a \int_0^{\phi(x)} \int_0^{\psi(x,y)} yz \, dz \, dy \, dx.
\]
A straightforward computation gives an answer of \( \frac{1}{15} abc^2 \).

Another Solution. We return to Problem 3 but this time carry out the integration in a different order. The same solid is now projected onto the \( yz \)-plane. In terms of \( y \) and \( z \) the curved surface has equation
\[
x = \Psi(y, z) = \frac{a}{bc} \sqrt{b^2 c^2 - c^2 y^2 - b^2 z^2}.
\]
This surface intersects the \( yz \)-plane in the curve
\[
y = \Phi(z) = \frac{b}{c} \sqrt{c^2 - z^2}.
\]
We can take
\[
\Omega_{yz} : 0 \leq z \leq c, 0 \leq y \leq \Phi(z)
\]
as the base region and characterize \( T \) as the set of all \((x, y, z)\) with
\[
0 \leq z \leq c, 0 \leq y \leq \Phi(z), 0 \leq x \leq \Psi(y, z).
\]
This leads to the repeated integral
\[
\int_0^c \int_0^{\Phi(z)} \int_0^{\psi(y,z)} yz \, dx \, dy \, dz
\]
which also gives \( \frac{1}{15} abc^2 \).

Problem 4. Use triple integration to find the volume of the solid \( T \) bounded above by the parabolic cylinder \( z = 4 - y^2 \) and bounded below by the elliptic paraboloid \( z = x^2 + 3 y^2 \).

Solution. Solving the two equations simultaneously, we have
\[4 - y^2 = x^2 + 3y^2 \text{ and thus } x^2 + 4y^2 = 4.\]

This tells us that the two surfaces intersect in a space curve that lies along the elliptic cylinder \(x^2 + 4y^2 = 4\). The projection of this intersection onto the \(xy\)-plane is the ellipse \(x^2 + 4y^2 = 4\).

The projection of \(T\) onto the \(xy\)-plane is the region
\[\Omega_{xy}: -2 \leq x \leq 2, -\frac{1}{2}\sqrt{4 - x^2} \leq y \leq \frac{1}{2}\sqrt{4 - x^2}.\]

The solid \(T\) is then the set of all \((x, y, z)\) with
\[-2 \leq x \leq 2, -\frac{1}{2}\sqrt{4 - x^2} \leq y \leq \frac{1}{2}\sqrt{4 - x^2}, x^2 + 3y^2 \leq z \leq 4 - y^2.\]

Its volume is given by
\[V = \iiint_{\Omega_{xy}} \int_{x^2 + 3y^2}^{4 - y^2} dz \, dy \, dx = 4\pi.\]

**CYLINDRICAL COORDINATES**

**Introduction to Cylindrical Coordinates**

The first two coordinates, \(r\) and \(\theta\), of the cylindrical coordinates \((r, \theta, z)\) of a point in \(xyz\)-space are the usual plane polar coordinates, except that \(r\) is taken as nonnegative and \(\theta\) is restricted to the interval \([0, 2\pi]\). The third coordinate is the third rectangular coordinate \(z\). In rectangular coordinates the coordinate surfaces

\[x = x_0, y = y_0, z = z_0\]

are three mutually perpendicular planes. In cylindrical coordinates the coordinate surfaces take the form

\[r = r_0, \theta = \theta_0, z = z_0.\]

The first surface is a circular cylinder of radius \(r_0\). The central axis of the cylinder is the \(z\)-axis. The surface \(\theta = \theta_0\) is a vertical half-plane hinged at the \(z\)-axis. The plane stands at an angle of \(\theta_0\) radians from the positive \(x\)-axis. The last coordinate surface is the plane \(z = z_0\) of rectangular coordinates.

The \(xyz\)-solids easiest to describe in cylindrical coordinates are the **cylindrical wedges**. A wedge consists of all points \((x, y, z)\) that have cylindrical coordinates \((r, \theta, z)\) in the box

\[\Pi: a_1 \leq r \leq a_2, b_1 \leq \theta \leq b_2, c_1 \leq z \leq c_2.\]
Evaluating Triple Integrals Using Cylindrical Coordinates

Suppose that $T$ is some basic solid in $xyz$-space, not necessarily a wedge. If $T$ is the set of all $(x, y, z)$ with cylindrical coordinates in some basic solid $S$ in $r\theta z$-space, then

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iiint_S f(r \cos \theta, r \sin \theta, z) \, r \, dr \, d\theta \, dz.$$  

**Derivation.** We will carry out the argument on the assumption that $T$ is projectable onto some basic region $\Omega_{xy}$ of the $xy$-plane. (It is for such solids that the formula is most useful.) $T$ has some lower boundary $z = \psi_1(x, y)$ and some upper boundary $z = \psi_2(x, y)$. $T$ is then the set of all $(x, y, z)$ with

$$(x, y) \in \Omega_{xy} \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y).$$

The region $\Omega_{xy}$ has polar coordinates in some set $\Omega_{r\theta}$ (which we assume is a basic region). Then $S$ is the set of all $(r, \theta, z)$ with

$$(r, \theta) \in \Omega_{r\theta} \text{ and } \psi_1(r \cos \theta, r \sin \theta) \leq z \leq \psi_2(r \cos \theta, r \sin \theta).$$

Therefore

$$\iiint_T f(x, y, z) \, dx \, dy \, dz = \iint_{\Omega_{xy}} \left( \int_{\psi_1(x, y)}^{\psi_2(x, y)} f(x, y, z) \, dz \right) \, dx \, dy$$

$$= \iint_{\Omega_{r\theta}} \left( \int_{\psi_1(r \cos \theta, r \sin \theta)}^{\psi_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \right) \, r \, dr \, d\theta$$

$$= \iiint_S f(r \cos \theta, r \sin \theta, z) \, r \, dr \, d\theta \, dz.$$  

**Volume Formula**

If $f(x, y, z) = 1$ for all $(x, y, z)$ in $T$, then the formula reduces to

$$\iiint_T dx \, dy \, dz = \iiint_S r \, dr \, d\theta \, dz.$$  

The triple integral on the left is the volume of $T$. In summary, if $T$ is a basic solid in $xyz$-space and the cylindrical coordinates of $T$ constitute a basic solid $S$ in $r\theta z$-space, then the volume of $T$ is given by the formula

$$V = \iiint_S r \, dr \, d\theta \, dz.$$  

**Calculations**

Cylindrical coordinates are particularly useful when an axis of symmetry is present. The axis of symmetry is then taken as the $z$-axis.

**Problem 1.** Find the mass of a solid cylinder $T$ of radius $R$ and height $h$ given that the density varies directly with the distance from the axis of the cylinder.
Solution. Place the cylinder \( T \) on the \( xy \)-plane so that the axis of \( T \) coincides with the \( z \)-axis. The density function then takes the form \( \lambda(x, y, z) = k\sqrt{x^2 + y^2} \) and \( T \) consists of all points \((x, y, z)\) with cylindrical coordinates \((r, \theta, z)\) in the set

\[
S : 0 \leq r \leq R, 0 \leq \theta \leq 2\pi, 0 \leq z \leq h.
\]

Therefore

\[
M = \iiint_T k\sqrt{x^2 + y^2} \, dx \, dy \, dz = \iiint_S (kr) \, r \, dr \, d\theta \, dz
\]

\[
= k \int_0^R \int_0^{2\pi} \int_0^h r^2 \, dr \, d\theta \, dz = \frac{2}{3}k\pi R^3 h.
\]

Problem 2. Use cylindrical coordinates to find the volume of the solid \( T \) bounded above by the plane \( z = y \) and below by the paraboloid \( z = x^2 + y^2 \).

Solution. In cylindrical coordinates the plane has equation \( z = r \sin \theta \) and the paraboloid has equation \( z = r^2 \). Solving these two equations simultaneously, we have \( r = \sin \theta \). This tells us that the two surfaces intersect in a space curve that lies along the circular cylinder \( r = \sin \theta \). The projection of this intersection onto the \( xy \)-plane is the circle with polar equation \( r = \sin \theta \). The base region \( \Omega_{xy} \) is thus the set of all \((x, y)\) with polar coordinates in the set

\[
0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta.
\]

\( T \) itself is the set of all \((x, y, z)\) with cylindrical coordinates in the set

\[
S : 0 \leq \theta \leq \pi, 0 \leq r \leq \sin \theta, r^2 \leq z \leq r \sin \theta.
\]

Therefore

\[
V = \iiint_T dxdydz = \iiint_S r \, dr \, d\theta \, dz
\]

\[
= \int_0^\pi \int_0^\sin \theta \int_0^{r \sin \theta} r \, dz \, dr \, d\theta = \int_0^\pi \int_0^{\sin \theta} (r^2 \sin \theta - r^3) \, dr \, d\theta
\]

\[
= \frac{1}{12} \int_0^\pi \sin^4 \theta d\theta = \frac{\pi}{32}.
\]

Problem 3. Locate the centroid of the solid in Problem 2.

Solution. Since \( T \) is symmetric about the \( yz \)-plane, we see that \( \bar{x} = 0 \). To get \( \bar{y} \) we begin as usual:

\[
\bar{y}V = \iiint_T y \, dx \, dy \, dz = \iiint_S (r \sin \theta) r \, dr \, d\theta \, dz
\]

\[
= \int_0^\pi \int_0^{\sin \theta} \int_0^{r \sin \theta} r^2 \sin \theta \, dz \, dr \, d\theta = \int_0^\pi \int_0^{\sin \theta} (r^3 \sin^2 \theta - r^4 \sin \theta) \, dr \, d\theta
\]

\[
= \int_0^\pi \frac{1}{20} \sin^6 \theta d\theta = \frac{\pi}{64}.
\]

Since \( V = \frac{\pi}{32} \), we have \( \bar{y} = \frac{1}{2} \). Now for \( \bar{z} \):

\[
\bar{z}V = \iiint_T z \, dx \, dy \, dz = \iiint_S zr \, dr \, d\theta \, dz = \int_0^\pi \int_0^{\sin \theta} \int_0^{r \sin \theta} zr \, dz \, dr \, d\theta = \int_0^\pi \frac{1}{24} \sin^6 \theta d\theta = \frac{5\pi}{384}.
\]

Division by \( V = \frac{\pi}{32} \) gives \( \bar{z} = \frac{5}{12} \). The centroid is thus the point \((0, \frac{1}{2}, \frac{5}{12})\).
THE TRIPLE INTEGRAL AS THE LIMIT OF RIEMANN SUMS; SPHERICAL COORDINATES

The Triple Integral as the Limit of Riemann Sums

We have seen how single integrals and double integrals can be obtained as limits of Riemann sums. The same holds true for triple integrals.

Start with a basic solid \( T \) in \( xyz \)-space and decompose it into a finite number of basic solids \( T_1, \ldots, T_N \). If \( f \) is continuous on \( T \), then \( f \) is continuous on each \( T_i \). From each \( T_i \) pick an arbitrary point \((x^*_i, y^*_i, z^*_i)\) and form the Riemann sum

\[
\sum_{i=1}^{N} f(x^*_i, y^*_i, z^*_i) \text{(volume of } T_i) \text{.}
\]

The triple integral over \( T \) is the limit of such sums; namely, given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, if the diameters of the \( T_i \) are all less than \( \delta \), then

\[
\left| \sum_{i=1}^{N} f(x^*_i, y^*_i, z^*_i) \text{(volume of } T_i) - \iiint_{T} f(x, y, z) \, dx \, dy \, dz \right| < \epsilon
\]

no matter how the \((x^*_i, y^*_i, z^*_i)\) are chosen within the \( T_i \). We express this by writing

\[
\iiint_{T} f(x, y, z) \, dx \, dy \, dz = \lim_{\text{diam } T_i \to 0} \sum_{i=1}^{N} f(x^*_i, y^*_i, z^*_i) \text{(volume of } T_i) .
\]

Introduction to Spherical Coordinates

The spherical coordinates \((\rho, \theta, \phi)\) of a point \( P \) in \( xyz \)-space have the following meaning. The first coordinate, \( \rho \), is the distance from the origin; thus \( \rho \geq 0 \). The second coordinate, the angle \( \theta \), is the longitude; \( \theta \) ranges from 0 to \( 2\pi \). The third coordinate, the angle marked \( \phi \), ranges only from 0 to \( \pi \) is the colatitude, or more simply the polar angle. (The complement of \( \phi \) would be the latitude on a globe.)

The coordinate surfaces \( \rho = \rho_0 \), \( \theta = \theta_0 \), \( \phi = \phi_0 \) have the following meaning. The surface \( \rho = \rho_0 \) is a sphere; the radius is \( \rho_0 \) and the center is the origin. The second surface, \( \theta = \theta_0 \), is the same as in cylindrical coordinates: the vertical half-plane hinged at the \( z \)-axis and standing at an angle of \( \theta_0 \) radians from the positive \( x \)-axis. The surface \( \phi = \phi_0 \) requires detailed explanation. If \( 0 < \phi_0 < \frac{\pi}{2} \) or \( \frac{\pi}{2} < \phi_0 < \pi \), the surface is a nappe of a cone; it is generated by revolving about the \( z \)-axis any ray that emerges from the origin at an angle of \( \phi_0 \) radians from the positive \( z \)-axis. The surface \( \phi = \frac{\pi}{2} \) is the \( xy \)-plane. (The nappe of the cone has opened up completely.) The equation \( \phi = 0 \) gives the positive \( z \)-axis, and the equation \( \phi = \pi \) gives the negative \( z \)-axis. (When \( \phi = 0 \) or \( \phi = \pi \), the nappe of the cone has closed up completely.)

Rectangular coordinates \((x, y, z)\) are related to spherical coordinates \((\rho, \theta, \phi)\) by the following equations:

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
\]

Conversely, with obvious exclusions, we have

\[
\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = \frac{y}{x}, \quad \cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.
\]

March 17, 2001
The Volume of a Spherical Wedge

A spherical wedge $W$ consists of all points $(x, y, z)$ that have spherical coordinates in the box

$$\Pi : a_1 \leq \rho \leq a_2, b_1 \leq \theta \leq b_2, c_1 \leq \phi \leq c_2.$$ 

The volume of this wedge is given by the formula

$$V = \iiint_{\Pi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$ 

Proof. Note first that $W$ is a solid of revolution. One way to obtain $W$ is to rotate the $\theta = b_1$ face of $W$, call it $\Omega$, about the $z$-axis for $b_2 - b_1$ radians. On that face $\rho$ and $\alpha = \frac{\pi}{2} - \phi$ play the role of polar coordinates. The face $\Omega$ is the set of all $(z, X)$ with polar coordinates $(\rho, \alpha)$ in the set $\Gamma : a_1 \leq \rho \leq a_2, \frac{\pi}{2} - c_2 \leq \alpha \leq \frac{\pi}{2} - c_1$. The centroid of $\Omega$ is at a distance $\bar{X}$ from the $z$-axis where

$$\bar{X}(\text{area of } \Omega) = \iint_{\Omega} X \, dX \, dz = \iint_{\Gamma} \rho^2 \cos \alpha \, d\rho \, d\alpha = (\int_{a_1}^{a_2} \rho^2 \, d\rho) (\int_{\frac{\pi}{2} - c_2}^{\frac{\pi}{2} - c_1} \cos \alpha \, d\alpha) = (\int_{a_1}^{a_2} \rho^2 \, d\rho) (\int_{c_1}^{c_2} \sin \phi \, d\phi).$$

As the face $\Omega$ is rotated from $\theta = b_1$ to $\theta = b_2$, the centroid travels through a circular arc of length

$$s = (b_2 - b_1)\bar{X} = (b_2 - b_1) \frac{1}{\text{area of } \Omega} (\int_{a_1}^{a_2} \rho^2 \, d\rho) (\int_{c_1}^{c_2} \sin \phi \, d\phi).$$

From Pappus formula we know that

$$\text{the volume of } W = s(\text{area of } \Omega) = (b_2 - b_1) (\int_{a_1}^{a_2} \rho^2 \, d\rho) (\int_{c_1}^{c_2} \sin \phi \, d\phi) = (\int_{b_1}^{b_2} d\theta) (\int_{a_1}^{a_2} \rho^2 \, d\rho) (\int_{c_1}^{c_2} \sin \phi \, d\phi) = \iiint_{\Pi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

Evaluating Triple Integrals Using Spherical Coordinates

Suppose that $T$ is a basic solid in $xyz$-space with spherical coordinates in some basic solid $S$ of $\rho \theta \phi$-space. Then

$$\iiint_{T} f(x, y, z) \, dx \, dy \, dz = \iiint_{S} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$ 

Derivation. Assume first that $T$ is a spherical wedge $W$. $S$ is then a box $\Pi$. Now decompose $\Pi$ into $N$ boxes $\Pi_1, \ldots, \Pi_N$. This induces a subdivision of $W$ into $N$ spherical wedges $W_1, \ldots, W_N$. Writing $F(\rho, \theta, \phi)$ for $f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$ to save space, we have

$$\iiint_{\Pi} F(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \sum_{i=1}^{N} \iiint_{\Pi_i} F(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$
THE TRIPLE INTEGRAL AS THE LIMIT OF RIEMANN SUMS; SPHERICAL COORDINATES

\[ \sum_{i=1}^{N} F(\rho_i^*, \theta_i^*, \phi_i^*) \int \int \int_{\Pi_i} \rho^2 \sin \phi \, d\rho d\theta d\phi = \sum_{i=1}^{N} F(\rho_i^*, \theta_i^*, \phi_i^*) \text{(volume of } W_i) = \sum_{i=1}^{N} f(x_i^*, y_i^*, z_i^*) \text{(volume of } W_i). \]

Here

\[ \begin{align*}
x_i^* &= \rho_i^* \sin \phi_i^* \cos \theta_i^* \\
y_i^* &= \rho_i^* \sin \phi_i^* \sin \theta_i^* \\
z_i^* &= \rho_i^* \cos \phi_i^*;
\end{align*} \]

and this expression is a Riemann sum for

\[ \iiint_T f(x, y, z) \, dxdydz \]

and, as such, will differ from that integral by less than any preassigned positive number \( \epsilon \) provided only that the diameters of all the \( W_i \) are sufficiently small. This we can guarantee by making the diameters of all the \( \Pi_i \) sufficiently small.

This verifies the formula for the case where \( T \) is a spherical wedge.

**Volume Formula**

If \( f(x, y, z) = 1 \) for all \( (x, y, z) \) in \( T \), then the change of variables formula reduces to

\[ \iiint_T dxdydz = \iiint_S \rho^2 \sin \phi d\rho d\theta d\phi. \]

The integral on the left is the volume of \( T \). It follows that the volume of \( T \) is given by the formula

\[ V = \iiint_S \rho^2 \sin \phi d\rho d\theta d\phi. \]

Spherical coordinates are commonly used in applications where there is a center of symmetry. The center of symmetry is then taken as the origin.

**Calculations**

**Problem 1.** Calculate the mass \( M \) of a solid ball of radius 1 given that the density varies directly with the square of the distance from the center of the ball.

**Solution.** Center the ball at the origin. The ball, call it \( T \), is now the set of all \( (x, y, z) \) with spherical coordinates \( (\rho, \theta, \phi) \) in the box

\[ S : 0 \leq \rho \leq 1, \, 0 \leq \theta \leq 2\pi, \, 0 \leq \phi \leq \pi. \]

Therefore

\[ M = \iiint_T k(x^2 + y^2 + z^2) \, dxdydz \]
\[ \begin{align*}
&= \iiint_S (k \rho^2) \rho^2 \sin \phi d\rho d\theta d\phi \\
&= k \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^4 \sin \phi d\rho d\theta d\phi \\
&= k \left( \int_0^\pi \sin \phi d\phi \right) \left( \int_0^{2\pi} d\theta \right) \left( \int_0^1 \rho^4 d\rho \right) \\
&= k(2)(2\pi)(\frac{1}{5}) = \frac{4}{5} k\pi.
\end{align*} \]

**Problem 2.** Find the volume of the solid \( T \) enclosed by the surface

\[(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2).\]

**Solution.** In spherical coordinates the bounding surface takes the form

\[ \rho = 2 \sin^2 \phi \cos \phi. \]

This equation places no restriction on \( \theta \); thus \( \theta \) can range from 0 to \( 2\pi \). Since \( \rho \) remains nonnegative, \( \phi \) can range only from 0 to \( \frac{\pi}{2} \). Thus the solid \( T \) is the set of all \((x, y, z)\) with spherical coordinates \((\rho, \theta, \phi)\) in the set

\[ S : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \rho \leq 2 \sin^2 \phi \cos \phi. \]

The rest is straightforward:

\[ V = \iiint_T dx dy dz = \iiint_S \rho^2 \sin \phi d\rho d\theta d\phi = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\sin^2 \phi \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta \]
\[ = \int_0^{2\pi} \int_0^{\pi/2} \frac{8}{3} \sin^7 \phi \cos^3 \phi d\phi d\theta = \frac{8}{3} \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\pi/2} (\sin^7 \phi \cos \phi - \sin^9 \phi \cos \phi) d\phi \right) \]
\[ = \frac{2\pi}{15}. \]