Mathematical Experiment 6

Cyclic Generators of $\mathbb{Z}_n \times \mathbb{Z}_k$

1. Let $\mathbb{Z}_k$ denote the additive group of integers modulo $k$. For integers $n$ and $k$, the Cartesian product $\mathbb{Z}_n \times \mathbb{Z}_k$ can be made into a group with respect to the componentwise group operation. An element $(r, s)$ in $\mathbb{Z}_n \times \mathbb{Z}_k$ is called a cyclic generator if the cyclic subgroup generated by $(r, s)$ coincides with $\mathbb{Z}_n \times \mathbb{Z}_k$ itself.

a. Is $(3,11)$ a cyclic generator of $\mathbb{Z}_{10} \times \mathbb{Z}_{12}$

b. Which of the following is a cyclic generator of $\mathbb{Z}_{10} \times \mathbb{Z}_{21}$: $(1,1)$, $(2,2)$, $(3,4)$, $(7,9)$, $(9,10)$?

c. Which elements in $\mathbb{Z}_4 \times \mathbb{Z}_5$ are cyclic generators?

d. Is $\mathbb{Z}_4 \times \mathbb{Z}_4$ a cyclic group?

e. What is the necessary and sufficient condition on $(n, k)$ that $\mathbb{Z}_n \times \mathbb{Z}_k$ is a cyclic group? What is the necessary and sufficient condition that $(r, s)$ is a cyclic generator of $\mathbb{Z}_n \times \mathbb{Z}_k$?

Sliding Triangle

2. Suppose that one vertex of a rigid regular triangle slides along the $x$-axis while the other vertex slides along the $y$-axis. Draw the locus of the third vertex. Refine your presentation so that each time [F8] is pressed, the triangle moves to a new position. Repeat the same procedure, replacing the regular triangle by an arbitrary triangle.

Moving Tangent and Normal

3. Consider the ellipse $\gamma(t) = [5 \cos t, 3 \sin t]$, $t \in [0, 2\pi]$.

a. Compute the velocity vector $\gamma'(t)$. 

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b. Compute the normal vector to the ellipse.

c. Find the loci of \( \gamma \).

d. Set up the demo so each time [F8] is press, the tangent and the normal to \((x_k, y_k)\) on the ellipse as well as the line segments joining \((x_k, y_k)\) to the loci, move to a new position corresponding to \((x_{k+1}, y_{k+1})\).

Radius of Curvature

4. For a smooth parametric curve \( \gamma(t) = (x(t), y(t)) \), the quantity

\[
\kappa = \frac{|x^0_0 y^0_0 - x^0_0 y^0_0|}{\left[(x^0_0)^2 + (y^0_0)^2\right]^{3/2}}
\]

is known as the curvature and \( \rho = 1/\kappa \) is known as the radius of curvature. If \( N \) denotes the unit normal to \( \gamma \), then the center of curvature is given by \( \gamma + \rho N \). Let \( t_0, t_1, \ldots, t_{100} \) be 101 equally spaced point in \([0, 2\pi]\). Let \( \gamma \) be the ellipse

\[
\gamma(t) = [5 \cos t, 3 \sin t].
\]

Draw the line segments joining \( \gamma(t_k) \) with the center of curvature corresponding to \( \gamma(t_k) \) for \( k = 0, 1, 2, \ldots, 100 \).

Pappus' Theorem

5. Draw the figure illustrating Pappus' Theorem: If the six vertices of a hexagon lie alternately on two lines, the three points of intersection of pairs of opposite sides are collinear.

![Diagram of Pappus' Theorem](image)

Brianchon’s Theorem

6. Illustrate Brianchon’s Theorem: If a hexagon is circumscribed about a circle, its
three diagonals are concurrent.

Pascal’s “Mystic Hexagram” Theorem

7. Illustrate Pascal’s Theorem: The points $L$, $M$, $N$ of intersection of the three pairs of opposite sides $AB$ and $DE$, $BC$ and $EF$, $FA$ and $CD$ of a hexagon $ABCDEF$ inscribed in a circle lie on a line.