Mathematical Experiment 5

Velocity Vector
1. Let $\gamma$ be a curve in $\mathbb{R}^2$ given by $\gamma(t) = [\cos t, \sin t]$, $t \in [0, 2\pi]$. Let $t_0,t_1,...,t_{100}$ be equally spaced points on the interval $[0,2\pi]$.
   a. Compute the velocity vector $\gamma'(t)$.
   b. Draw the line segments joining $\gamma(t_k)$ with $\gamma(t_k) + \gamma'(t_k)$ for $k = 0, 1, ..., 100$.
   c. Repeat the same drawing for $\gamma(t) = [\frac{\cos t}{3+\sin t}, \frac{1-\sin t}{4-\cos t}]$, $t \in [0, 2\pi]$.

Determinant of a Tridiagonal Matrix
2. Let $M_n$ be an $n \times n$ tridiagonal matrix of the form

$$M_n = \begin{bmatrix}
x & 1 & & & \\
1 & x & 1 & & \\
& 1 & x & & \\
& & \ddots & \ddots & \\
& & & 1 & x \\
& & & & 1 & x
\end{bmatrix}.$$

   • Find a recursive formula satisfied by the determinant $d_n(x)$ of $M_n$.
   • Compute $d_n(x)$ for $n = 2, 3, ..., 12$.

Sequence of Hofstadter
3. Let $\{a_n\}$ be the sequence given by

$$a_1 = 1 = a_2$$
$$a_n = a_{n-a_{n-1}} + a_{n-a_n-2} \text{ for } n > 2.$$

1,1,2,3,3,4,5,6,6,6,8,8,8,10,9,10,11,11,12,12,12,12,16,14,14,16,16,16,16,16,20,17,17,...

List the first 100 terms of $a_n$.

Four Number Game
4. Take any four positive integers, e.g. 8, 17, 3, 107. Under each number form the absolute difference between it and the following number. Thus

\begin{align*}
8 & \quad 17 & \quad 3 & \quad 107 \\
9 & \quad 14 & \quad 104 & \quad 99
\end{align*}
To obtain the last difference the first number is considered to follow the fourth. Therefore \(99 = |107 - 8|\). Repeat the above steps. What can be concluded? (Reference: B. Freedman, The four number game, Scripta Mathematica (1948), 35-47)

Interesting Design

5.
   a. Draw a convex pentagon.
   b. Draw the line segments joining the successive midpoints to form another pentagon inside the given one.
   c. Repeat the above step on the newly formed pentagon 20 times.
   d. Do the same as in steps (a)-(c) except that the \(k\)th vertex \(v_k\) of the new pentagon is given by
      \[ v_k = 0.9w_k + 0.1w_{k+1} \pmod{5} \]
      for \(k = 0, 1, 2, 3, 4\).
   e. Repeat step (d), except replacing the pentagon by a regular triangle.
   f. By adding the reflection of the triangle obtained in step (e) we obtain a rhombus. By adding the rotations of the rhombus we obtain a hexagon. By adding several of the translations of the hexagon we obtain the final design.

Stars Inside a Star

6. Select any five points on the plane so they form the vertices of a convex pentagon. With these five points draw a “star” by joining line segments from the first point to the third point and then to the fifth, the second, the fourth and then the first point in that order. As a result, five more points at the intersections of the line segments are obtained. Repeat the process by drawing a smaller star with these five points as vertices, then a still smaller stars and so on. It can be proved that there is exactly one point inside all the nested stars. Can the coordinates of this unique point be described
in terms of those of the originally selected five point? (This problem remains open!)