Mathematical Experiment 13

Horner’s Method

1. To evaluate a polynomial function such as

\[ p(x) = 2 + 3x - 4x^2 + 5x^3 - x^4 + 7x^5 \]

simply write it in the form

\[ p(x) = 2 + x(3 + x(-4 + x(5 + x(-1 + 7x)))) \].

This way, instead of performing multiplication \(1 + 2 + 3 + 4 + 5 = 15\) times, we need to multiply only 5 times. To evaluate a polynomial such as

\[ p(x) = 3 + 4(x - 2) - 5(x - 2)^2 + 3(x - 2)^3 - 6(x - 2)^4, \]

rewrite it in the form

\[ p(x) = 3 + (x - 2)(4 + (x - 2)(-5 + (x - 2)(3 - 6(x - 2)))) \].

More generally, to evaluate a polynomial such as

\[ p(x) = 2 + 4(x - 3) + 5(x - 3)(x + 2) - 6(x - 3)(x + 2)(x - 7) + 8(x - 3)(x + 2)(x - 7)(x + 3), \]

just transform it into

\[ p(x) = 2 + (x - 3)(4 + (x + 2)(5 - (x - 7)(-6 + 8(x + 3)))) \].

Set up the spreadsheet to perform each of the above tasks.

Polynomial Expansion

2. 

- Find the expansion of

\[ p(x) = 3 + 4(x - 2) - 5(x - 2)^2 + 3(x - 2)^3 - 6(x - 2)^4. \]

- Find the expansion of

\[ p(x) = 2 + 4(x - 3) + 5(x - 3)(x + 2) - 6(x - 3)(x + 2)(x - 7) + 8(x - 3)(x + 2)(x - 7)(x + 3), \]

- Find the Taylor expansion of \( p(x) = 2 + 3x - 4x^2 + 5x^3 - x^4 + 7x^5 \) at \( x = 3 \).
Power Series Expansion

3. The product of two formal power series

\[ B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \]

and

\[ C(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \]

is the formal power series

\[ A(x) = B(x)C(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

with the coefficients \( a_k \) given by the formula

\[ a_k = b_0 c_k + b_1 c_{k-1} + b_2 c_{k-2} + \cdots + b_{k-1} c_1 + b_k c_0. \]

If the leading coefficient \( b_0 \) of \( B(x) \) equal 1, then the above formula gives

\[ c_k = a_k - b_1 c_{k-1} - b_2 c_{k-2} - \cdots - b_{k-1} c_1 - b_k c_0. \]

The power series expansion of a rational function of the form

\[ \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r}{1 + b_1 x + b_2 x^2 + \cdots + b_s x^s} \]

can be computed using this formula.

Arranged the spreadsheet to compute the power series expansion of such rational function.

- Find the power series expansion of \( \frac{2 + 3x + 7x^2 + x^6 - x^8}{1 - x + 4x^2 + x^3 + 2x^4 - x^5 - x^6} \) up to \( x^{30} \).

\[
\begin{array}{cccc}
  2 & 2 \\
  3 & 5 \\
  7 & 4 \\
  0 & -18 \\
  0 & -43 \\
  0 & 17 \\
  1 & 207 \\
  0 & 227 \\
\end{array}
\]
Let \( f(x) = \frac{2+3x+7x^2+x^6-x^8}{1-x+4x^2+x^3+2x^4-x^5-x^6} \). Find \( f^{(n)}(0) \) for \( n = 0, 1, 2, \cdots, 30 \).

Express \( \frac{2+3x+7x^2+x^6-x^8}{1-x+4x^2+x^3+2x^4-x^5-x^6} \) in the form \( \sum_{n=0}^{\infty} a_n(x-2)^n \).

Let \( f(x) = \frac{2+3x+7x^2+x^6-x^8}{1-x+4x^2+x^3+2x^4-x^5-x^6} \). Find \( f^{(n)}(3) \) for \( n = 0, 1, 2, \cdots, 20 \).

Factorial Representation

4.

Express \( p(x) = 2x^4 - 7x^3 + 3x^2 + 4x + 5 \) in the form

\[
p(x) = a(x-3)(x+1)(x-4)(x+3)+b(x-3)(x+1)(x-4)+c(x-3)(x-1)+d(x-3)+e.
\]

Express \( p(x) = 2x^4 - 7x^3 + 3x^2 + 4x + 5 \) in the form

\[
p(x) = a(x-1)(x-2)(x-3)(x-4)+b(x-1)(x-2)(x-3)+c(x-1)(x-2)+d(x-1)+e.
\]