Equality holds here if and only if $Q=Q_*$. In a similar way one obtains this inequality for $x<x^*$. Thus the minimum time property of the path $P_1Q_*\cup Q_*P_2$ is proved.

References

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A GEOMETRIC PROOF OF THE EQUIVALENCE OF FERMAT'S PRINCIPLE AND SNELL'S LAW

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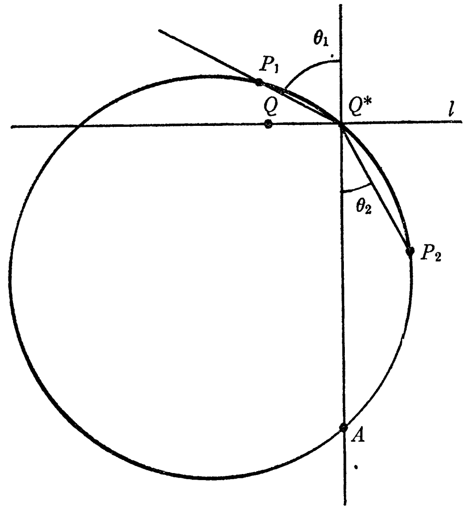
Let l be the line separating the media, and let $P_1Q^*P_2$ be the actual path of the ray according to Snell's Law, where $\sin\theta_1/\sin\theta_2=v_1/v_2$. We wish to show that for any other point Q on the line

$$P_1Q/v_1+P_2Q/v_2>P_1Q^*/v_1+P_2Q^*/v_2,$$

so that the time taken for traversing the actual path is a minimum. Draw the circle through the points P_1, Q^* and P_2, and let the perpendicular to l through Q^* intersect this circle again at the point A. Then we note that $AP_1=2R\sin\theta_1$, and $AP_2=2R\sin\theta_2$, where R is the radius of the circle $P_1Q^*P_2$. Therefore $AP_1/AP_2=v_1/v_2$; that is, $AP_1=k/v_2$ and $AP_2=k/v_1$, where k is a constant.



Applying the theorem of Ptolemy to the four concyclic points P_1, Q^*, P_2 and A, we have the equality

$$P_1P_2\cdot AQ^*=P_1Q^*\cdot AP_2+P_2Q^*\cdot AP_1,$$

whereas if Q is any point $\neq Q^*$ on the line l, the extension of the Ptolemy theorem which arises naturally by inverting the triangle inequality [1] gives the inequality

$$P_1P_2 \cdot AQ < P_1Q \cdot AP_2 + P_2Q \cdot AP_1.$$

If we substitute for AP_1 and AP_2, we obtain the equality

$$k(P_1Q^*/v_1 + P_2Q^*/v_2) = P_1P_2 \cdot AQ^*,$$

and the inequality

$$k(P_1Q/v_1 + P_2Q/v_2) > P_1P_2 \cdot AQ.$$

Hence the Fermat principle of minimum time is established for a ray which satisfies Snell's Law, since $AQ > AQ^*$.

Reference

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ON THE RIEMANN INTEGRAL IN TWO DIMENSIONS

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A subdivision of an interval I consists of a finite number of intervals having at most edges or vertices in common, whose union is I.

In dealing with Riemann type of integrals on a two dimensional compact interval I, the norm of a subdivision σ of I is available in at least two manners: (a) If $|J|_A$ is the area of the interval J, then the norm $|\sigma|_A$ is the maximum $|J|_A$ for any J of σ; (b) if $|J|_S$ = maximum side length of J, then the norm $|\sigma|_S$ is the maximum of $|J|_S$ for all J of σ. A limit taken as $|\sigma|_S \to 0$ is weaker than a limit as $|\sigma|_A \to 0$, since any statement true for all σ such that $|\sigma|_A < d$ will be true for all σ such that $|\sigma|_S < \sqrt{d}$ (see [1], pp. 102–103).

It is well known that the norm customary in the theory of the Riemann integral in two dimensions is the norm $|\sigma|_S$, but no explanation of this fact is given. The avoidance of $|\sigma|_A$ can be justified as follows. If the norm $|\sigma|_A$ is used, then the theorem "*A continuous function on I is Riemann integrable on I*" will not be true. To prove this, let I be the unit square and let σ_n be the subdivision formed by n equidistant parallels to the x-axis. For instance, σ_3 is constituted by the segments (x, y) with $0 \leqq x \leqq 1$ and $y = 0, 1/3, 2/3, 1$. Consider now the function $f(x, y) = x$. If σ_n consists of the intervals $J_1, J_2, \cdots, J_i, \cdots, J_n$, then, by choosing $(\xi_i, \eta_i) \in J_i$ $(1 \leqq i \leqq n)$, we obtain the Riemann sum

$$S_{\sigma_n} = \sum_{i=1}^{n} f(\xi_i, \eta_i) \cdot |J_i|_A = \sum_{i=1}^{n} \xi_i |J_i|_A.$$

Now consider a number ξ such that $0 < \xi < 1$ and, for each $i \leqq n$ and for each